대기행렬 이론 개론

I. Basic Probability Theory

학습에 앞서

- 학습 목표
  - 대기행렬이론이 무엇인지 학습하고, 실생활 및 통신네트워크에서의 적용 분야를 학습한다.
  - 확률모형 및 확률과정의 분석에 필요한 기초 개념 및 기초 확률이론의 지식을 습득한다.

- 목차

Introduction

- Queueing Theory is mainly seen as a branch of applied probability theory. Its applications are in different fields, e.g. communication networks, computer systems, machine plants and so forth.
- The subject of queueing theory: consider a service center and a population of customers, which at some times enter the service center in order to obtain service.

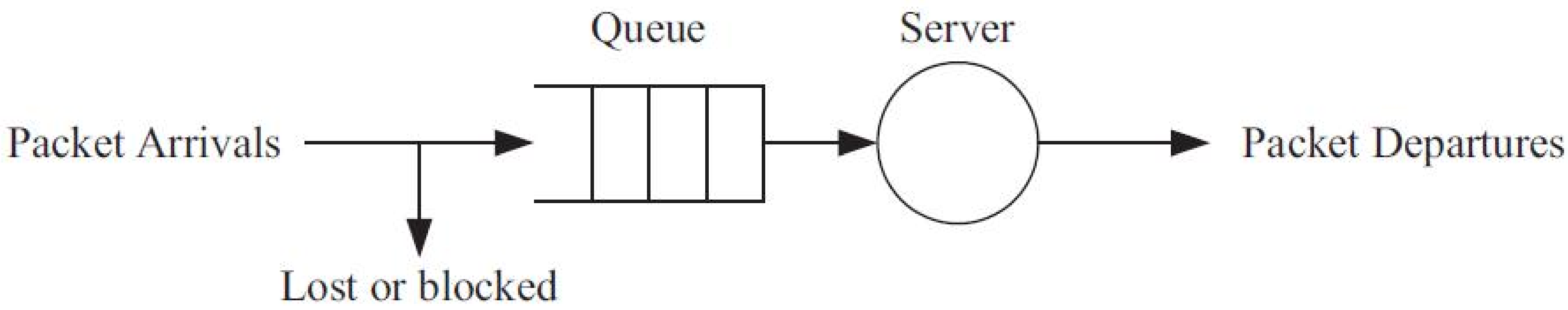
Examples of Queueing Theory (1/2)

- Examples of the use of queueing theory in networking
  - Determining the number of trunks in a central office in plain old telephone service (POTS)
  - Calculating end-to-end throughput in networks
- Other examples are:
  - Waiting to pay in the supermarket
  - Waiting to pay in the tollgate
  - Waiting at the telephone for information
- We may have the following questions:
  - What is the average waiting time of a customer?
  - How many customers are waiting on average
  - How long is the average service time

Examples of Queueing Theory (2/2)

- Queueing Theory tries to answer questions like e.g. the mean waiting time in the queue, the mean system response time (waiting time in the queue plus service times), mean utilization of the service facility, distribution of the number of customers in the queue, distribution of the number of customers in the system and so forth.

- A simplified queueing model for a delay/loss system



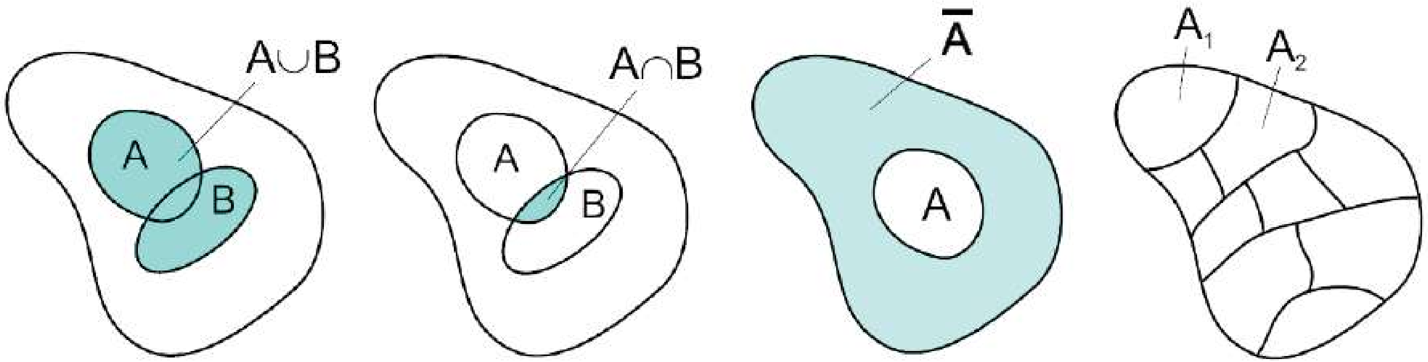
- Basic Notations

- Random experiment
  An experiment whose outcome cannot be determined in advance.
- Sample space ($\mathbf{S}$)
  The set of all possible outcomes of a random experiment
  Example 1. In tossing of a die we have $\mathbf{S} = \{1,2,3,4,5,6\}$.
  Example 2. The life-time of a bulb $\mathbf{S} = \{x \in \mathbf{R} \mid x > 0\}$.
- Event (A): A subset of sample space.
  An event is a subset of the sample space $\mathbf{S}$. An event is usually denoted by a capital letter $A, B, \cdots$. If the outcome of an experiment is a member of event A, we say that A has occurred.
  Example 1. The outcome of tossing a die is an even number: $A = \{2,4,6\} \subset \mathbf{S}$.
  Example 2. The life-time of a bulb is at least 3000 h: $A = \{x \in \mathbf{R} \mid x > 3000\} \subset \mathbf{S}$.
  Certain event: The whole sample space $\mathbf{S}$.
  Impossible event: Empty subset $\emptyset$ of $\mathbf{S}$.

Combining Events

- Union "A or B"
  $A \cup B = \{e \in \mathbf{S} \mid e \in A \text{ or } e \in B\}$
- Intersection (joint event) "A and B"
  $A \cap B = \{e \in \mathbf{S} \mid e \in A \text{ and } e \in B\}$
- Mutually exclusive
  Events A and B are mutually exclusive, if $A \cap B = \emptyset$.
- Complement "not A"
  $\overline{A} = \{e \in \mathbf{S} \mid e \notin A\}$
- Partition of the sample space A set of events $A_1, A_2, \cdots$ is a partition of the sample space $\mathbf{S}$

  1. The events are mutually exclusive, $A_i \cap A_j = \emptyset$, when $i \neq j$
  2. Together they cover the whole sample space, $\bigcup_i A_i = \mathbf{S}$.



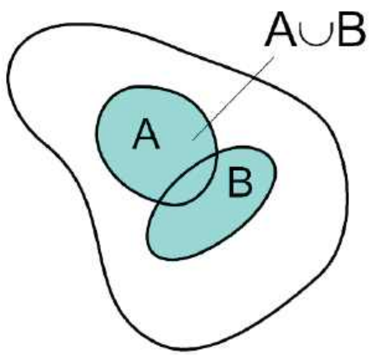
Probability

With each event A is associated the probability $P(A)$. Empirically, the probability $P(A)$ means the limiting value of the relative frequency $N(A)/N$ with which A occurs in a repeated experiment.

$$P(A) = \lim_{N\to\infty} \frac{N(A)}{N}$$

where N is the number of experiments and $N(A)$ is the number of occurrences of A.

Properties of probability

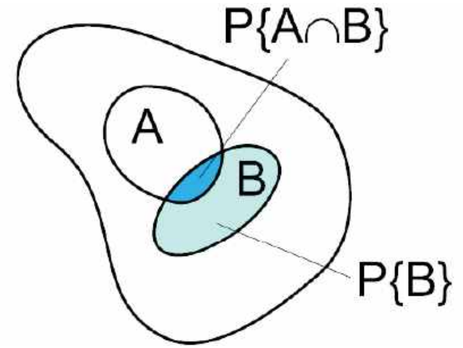


1. $0 \leq P(A) \leq 1$
2. $P(\mathbf{S}) = 1, P(\emptyset) = 0$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. If $A \cap B = 0$, then $P(A \cup B) = P(A) + P(B)$
   If $A_i \cap A_j = 0$, for $i \neq j$, then $P(\bigcup_i A_i) = P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$
5. $P(\overline{A}) = 1 - P(A)$
6. If $A \subseteq B$, then $P(A) \leq P(B)$

Conditional Probability

The probability of event A given that B has occurred.

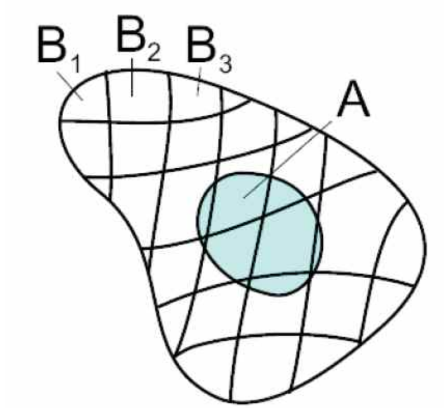
$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \Rightarrow \quad P(A \cap B) = P(A|B)P(B)$$



- Law of total probability

Let $\{B_1, \cdots, B_n\}$ be a complete set of mutually exclusive events, i.e. a partition of the sample space $\mathbf{S}$,

$$P(A) = \sum_i^n P(A \cap B_i) = \sum_{i=1}^n P(A|B_i)P(B_i)$$



Baye's formula

Let again $\{B_1, \cdots, B_n\}$ be a partition of the sample space. The problem is to calculate the probability of event B_i given that A has occurred.

$$P(B_i|A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)}$$

Bayes' formula enables us to calculate a conditional probability when we know the reverse conditional probabilities.

- Independence

Two events A and B are *independet* if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

For independent events holds

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A) \quad \text{“}B \text{ does not influence occurrence of } A\text{”}$$

Problem: Binary Symmetric Channel (1/2)

Problem (Binary Symmetric Channel) Consider a *discrete memoryless channel* used to transmit binary data. The channel is said to be discrete that it is designed to handle discrete message. It is memoryless in the sense that the channel output at any time depends only on the channel input at that time. Due to the unavoidable presence of noise on the channel, errors are made in the received binary data stream. The channel is described as follows:

- The *a priori* probability of sending binary symbols 0 and 1 is given by

$$P(X = x) = \begin{cases} p_0 & \text{if } x = 0 \\ p_1 & \text{if } x = 1 \end{cases}$$

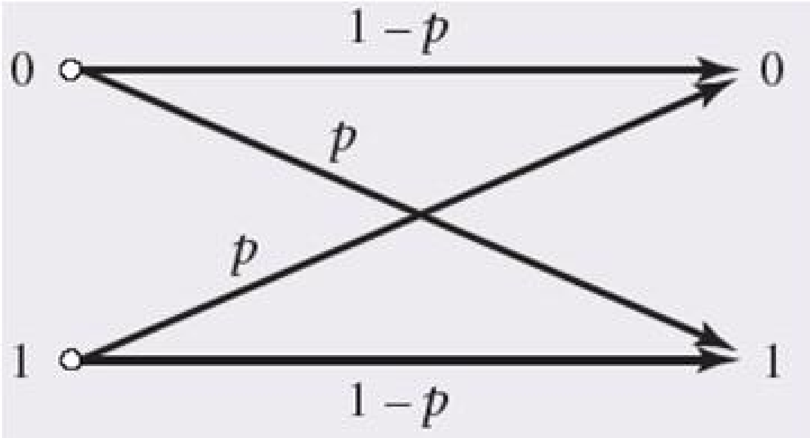
  where X is the random variable representing the transmitted symbol. Note that $p_0 + p_1 = 1$, so X is a Bernoulli random variable.
- The conditional probability of error is given by

$$P(Y = 1|X = 0) = P(Y = 0|X = 1) = p$$

  where Y is the random variable representing the received symbol. The conditional probability $P(Y = 0|X = 1)$ is the probability that symbol 0 is received given that symbol 1 was sent.

Determine the probability that the transmitter transmits the symbol 0 when the receiver receives the symbol 0. □

Problem: Binary Symmetric Channel (2/2)



Solution The probability of receiving symbol 0 is given by

$$\begin{aligned} P(Y=0) &= P(Y=0|X=0)P(X=0) + P(Y=0|X=1)P(X=1) \\ &= (1-p)p_0 + pp_1 \end{aligned}$$

Applying Bayes' rule, we obtain

$$\begin{aligned} P(X=0|Y=0) &= \frac{P(Y=0|X=0)P(X=0)}{P(Y=0)} \\ &= \frac{(1-p)p_0}{(1-p)p_0 + pp_1} \end{aligned}$$

□

2. Random Variable and Distributions

- Random Variable

A random variable (r.v.) is a function that assigns a real value to each element in **S**. Random variables are denoted by capitals, X, Y, etc.

Example 2.2. The number of heads in tossing coin rather than the sequence of heads/tails
A real-valued random variable X is a mapping
$X: \quad \mathbf{S} \mapsto \mathbf{R}$
which associates the real number $X(e)$ to each outcome $e \in \mathbf{S}$. □

- Distribution Function, Cumulative Distribution Function (CDF)

$$F(x) = P(X \leq x)$$

– Probability of an interval

$$P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$$

Probability Density Function

If X is a continuous r.v., the probability density function (pdf) is

$$f(x) = \frac{dF(x)}{dx} = \lim_{dx \to 0} \frac{P(x < X \leq x + dx)}{dx}$$

Hence,

$$F(x) = \int_{-\infty}^{x} f(y)dy$$

If X is a discrete (finite or countably infinite) r.v., the probability mass function (pmf) is

$$p(x) = P(X = x)$$

Hence,

$$F(x) = \sum_{y \leq x} P(X = y)$$

Joint Distribution

Joint distribution function

$$F(x,y) = P(X \le x, Y \le y)$$
$$F_X(x) = \lim_{y \to \infty} F(x,y)$$

If X and Y are independent, $F(x,y) = F_X(x)F_Y(y)$

Joint probability density function

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$
$$f(x,y)dxdy = P(x < X \le x+dx, y < Y \le y+dy)$$
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$
$$P(X \in A, Y \in B) = \int_A \int_B f(x,y)dydx$$

If X and Y are independent, $f(x,y) = f_X(x)f_Y(y)$

3. Parameters of Distributions (1/2)

Expectation

The expected value of X, denoted by $E(X)$ or $\overline{X}$, is defined as

$$E(X) = \int_{-\infty}^{\infty} x dF(x)$$

where $dF(x)$ is the probability of the interval dx.
Continuous distribution: $E(X) = \int_{-\infty}^{\infty} x f(x) dx$
Discrete distribution: $E(X) = \sum_i x_i p_i$

The expected value of an arbitrary function of X, $g(X)$, with respect to the probability density function $f(x)$ is given by the inner product of f and g:

$$E(g(X)) = \int_x E(g(X)|X = x) dF(x) = \int_x g(x) dF(x)$$
$$E(X) = \int_x E(X|X = x) dF(x) = \int_x x dF(x)$$

Properties:

$E(cX) = cE(X) c$ is constant

Parameters of Distributions (2/2)

Variance

If a random variable X has the expected value (mean) $\mu = E(X)$, then the variance of X, denoted by $Var(X)$ or $V(X)$, is given by:

$$Var(X) = E\left[(X - E(X))^2\right] = E(X^2) - E^2(X)$$

This definition encompasses random variables that are discrete, continuous, or neither. It can be expanded as follows:

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2] \\
&= \mathrm{E}[X^2 - 2\mu X + \mu^2] \\
&= \mathrm{E}[X^2] - 2\mu\,\mathrm{E}[X] + \mu^2 \\
&= \mathrm{E}[X^2] - 2\mu^2 + \mu^2 \\
&= \mathrm{E}[X^2] - \mu^2 \\
&= \mathrm{E}[X^2] - \mathrm{E}[X]^2.
\end{aligned}$$

Properties:

$$Var(cX) \qquad = c^2 Var(X)$$

- Generating Functions

Let $P_k = P(X = k)$. Definition is

$$P_X(z) \triangleq E(z^X) = \sum_{k=0}^{\infty} z^k P_k = P_0 + P_1 z + P_2 z^2 + \cdots$$

Properties are

$$P_X(1) = 1 \qquad \frac{d}{dz} P_X(z)\Big|_{z=1} = E(X) \qquad \frac{d^2}{dz^2} P_X(z)\Big|_{z=1} = E\left[X(X-1)\right] = E(X^2) - E(X)$$

$$\begin{aligned} V(X) &= E(X^2) - E^2(X) \\ &= \frac{d^2}{dz^2} P_X(z)\Big|_{z=1} + \frac{d}{dz} P_X(z)\Big|_{z=1} \left\{ \frac{d}{dz} P_X(z)\Big|_{z=1} \right\}^2 \end{aligned}$$

Given $P(z)$,

$$P_1 = \frac{d}{dz} P(z)\Big|_{z=0} \qquad P_k = \frac{1}{k!} \frac{d^k}{dz^k} P(z)\Big|_{z=0}$$

Transforms – Problem (1/2)

Problem Determine the generating function, $P(z)$, from the following equations:

$$\lambda P_0 = \mu P_1$$
$$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}, \text{for } n = 1, 2, \cdots$$

Solution

$$\sum_{k=1}^{\infty} P_k z^k = \sum_{k=0}^{\infty} P_k z^k - P_0 = P(z) - P_0$$
$$\sum_{k=1}^{\infty} P_{k-1} z^k = z \sum_{k=1}^{\infty} P_{k-1} z^{k-1} = zP(z)$$
$$\sum_{k=0}^{\infty} P_{k+1} z^k = \frac{1}{z} \sum_{k=0}^{\infty} P_{k+1} z^{k+1} = \frac{1}{z}(P(z) - P_0)$$

$$\lambda P_0 z^0 = \mu P_1 z^0$$
$$(\lambda+\mu)P_1 z^1 = \lambda P_0 z^1 + \mu P_2 z^1$$
$$(\lambda+\mu)P_2 z^2 = \lambda P_1 z^2 + \mu P_3 z^2$$
$$\vdots = \vdots$$
$$\lambda \sum_{k=0}^{\infty} P_k z^k + \mu \sum_{k=1}^{\infty} P_k z^k = \lambda \sum_{k=1}^{\infty} P_{k-1} z^k + \mu \sum_{k=0}^{\infty} P_{k+1} z^k$$
$$\lambda P(z) + \mu(P(z) - P_0) = \lambda z P(z) + \mu z^{-1}(P(z) - P_0)$$

Transforms – Problem (2/2)

$$\lambda P(z) + \mu(P(z) - P_0) = \lambda z P(z) + \mu z^{-1}(P(z) - P_0)$$

$$P(z) = \frac{P_0 \mu (z^{-1} - 1)}{\frac{\mu}{z} - (\lambda + \mu) + \lambda z}$$

$$= \frac{\mu(1 - z)P_0}{\mu - (\lambda + \mu)z + \lambda z^2}$$

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$P(z) = \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)z}$$

$$= (1 - \lambda/\mu) \sum_{k=0}^{\infty} \left(\frac{\lambda z}{\mu}\right)^k$$

$$P_k = (1 - \rho)\rho^k, (\rho = \lambda/mu)$$

Laplace Transform

Laplace Transform

X is non-negative and a continuous variable. Assume $F(0) = 0$ and $F(x)$ has no discontinuous point.

Definition is

$$F_X^*(s) = \mathscr{L}(f(x)) = \int_0^\infty e^{-sx} f_X(x)dx = E(e^{-sX})$$

Properties are

$$F_X^*(0) = 1$$
$$\frac{d^k}{ds^k} F_X^*(s)\bigg|_{s=0} = (-1)^k E(X^k)$$

Laplace-Stieltjes Trasform

X is non-negative and a continuous variable. Assume $F(0) > 0$ or $F(x)$ has a discontinuous point.

Definition is

$$F_X^*(s) = \widetilde{\mathscr{L}}(F(x)) = \int_0^\infty e^{-sx} dF(x) = E(e^{-sX})$$

Summary

- Queueing Theory tries to answer questions like
  - the mean waiting time in the queue,
  - the mean system response time
  - mean utilization of the service
  - distribution of the number of customers

- Basic Probability Theory
  - Probability: Conditional Probability, Baye’s formula
  - Random Variable and Distributions
  - Parameters of Distributions: Expectation, Variance
  - Transforms
    - Generating Functions
    - Laplace Transform