대기행렬 이론 개론

II. Distribution Functions

학습에 앞서

- 학습 목표
  - Order statics 분포를 이해하고, 통신네트워크에서의 응용을 학습한다.
  - 지수 분포, 얼랑 분포, 감마 분포 함수들을 학습한다.
  - 지수 분포와 포아송 프로세스의 기본 개념 및 응용을 학습한다.

- 목차
  - 1. Order Statistics
  - 2. Common Distributions
  - 3. Poisson Process

1. Order Statistics (1/4)

Let $X_{(n)}$ denote the nth smallest of element of the sample X. This statistics is called the *order statistic* of order n.

For example, suppose that four numbers are observed or recorded, resulting in a sample of size $n = 4$. If the sample values are

$$6, 9, 3, 8,$$

The order statistics would be denoted

$$x_{(1)} = 3; \ \ x_{(2)} = 6; \ \ x_{(3)} = 8; \ \ x_{(4)} = 9$$

The **first order statistic** (or *smallest order statistic*) is always the *minimum* of the sample, that is,

$$X_{(1)} = \min\{X_1, \ldots, X_n\}$$

where, following a common convention, we use upper-case letters to refer to random variables, and lower-case letters (as above) to refer to their actual observed values.

Similarly, for a sample of size n, the n**th order statistic** (or *largest order statistic*) is the *maximum*, that is,

$$X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

Order Statistics (2/4)

Distribution of max and min of independent random variables

Let $X_1, X_2, \ldots, X_n$ be independent random variables.
Distributions of the maximum and the minimum are

$$\begin{aligned}
P(\max(X_1, \cdots, X_n) \leq x) &= P(X_1 \leq x, \cdots, X_n \leq x) \\
&= P(X_1 \leq x) \cdots P(X_n \leq x)(\text{independence}) \\
&= F_1(x) \ldots F_n(x) \\
P(\min(X_1, \cdots, X_n) > x) &= P(X_1 > x, \cdots, X_n > x) \\
&= P(X_1 > x) \cdots P(X_n > x)(\text{independence}) \\
&= G_1(x) \ldots G_n(x)
\end{aligned}$$

where $F_i(x)$ is the distribution function and $G_i(x)$ is the tail distribution.

Order Statistics (3/4)

Problem High system throughput can be achieved by exploiting multiuser diversity and prioritized scheduling users with high instantaneous channel conditions when a base station (BS) has channel state information (CSI). [11–13, 15]. There are K active users and each user experiences independent fading. The channel of user k is assumed to be a block Rayleigh fading model with the probability density function given by f_{γ_k}, where γ_k is the received signal-to-noise ratio (SNR) of user k and $\overline{\gamma_k}$ is the average received SNR of user k.

1. When all users feeds back their CSI to the BS, determine the average sum-rate.

2. When each user feeds back its CSI to the BS if and only if $\gamma_k \geq \gamma_{th}$, where assume that the users experience i.i.d. fading. Determine the average sum-rate.

Order Statistics (4/4)

Solution

$$C = \int_0^\infty \log_2(1+\gamma^*) f_{\gamma^*}(\gamma) d\gamma$$

where γ^* is the SNR of the selected user.

1. When all users feeds back their CSI,

$$f_{\gamma^*}(\gamma) = \sum_{k=1}^{K} f_{\gamma_k}(\gamma) \prod_{l=1, l\neq k}^{K} F_{\gamma_k}(\gamma)$$

2. When each user feeds back its CSI if and only if $\gamma_k \geq \gamma_{th}$,

$$F_{\gamma^*}(\gamma) = \sum_{k=1}^{K} \binom{K}{k} (F_\gamma(\gamma_{th}))^{K-k} \left(F_\gamma(\gamma) - F_\gamma(\gamma_{th})\right)^k$$

$$f_{\gamma^*}(\gamma) = \sum_{k=1}^{K} \binom{K}{k} (F_\gamma(\gamma_{th}))^{K-k} k f_\gamma(\gamma) \left(F_\gamma(\gamma) - F_\gamma(\gamma_{th})\right)^{k-1}, \quad \gamma > \gamma_{th}$$

$$f_{\gamma^*}(\gamma) = \left(F_\gamma(\gamma_{th})\right)^{K-1} f_\gamma(\gamma), \quad \gamma \leq \gamma_{th}$$

2. Common Distributions (1/2)

- Probability Distribution of Discrete r.v.

Distribution	Description	pmf	Mean	Variance
Bern(p)	Bernoulli distribution	$P(X=1)=p,$ $P(X=0)=1-p$	p p	pq pq
Bin(n,p)	Binomial distribution	$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$	np	npq
Geom(p)	Geometric distribution: number of trials until the outcome is the front side (coin tossing)	$P(X=k)=(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{q}{p^2}$
Poisson(λ)	Poisson distribution	$P(X=k)=\frac{\lambda^k \exp^{-\lambda}}{k!}$	λ	λ

Common Distributions (2/2)

- Probability Distribution of Continuous r.v.

Distribution	Description	pdf	Mean	Variance
Unif(a,b)	Uniform distribution	$f(x) = \frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exp(λ)	Exponential distribution	$f(x) = \lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Erlang(n,λ)	Erlang distribution of order n	$f(x) = \frac{(\lambda x)^{n-1}}{(n-1)!}\lambda e^{-\lambda x} \quad x \geq 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$

Exponential Distribution, Exp(λ) (1/2)

The density of an exponential distribution with parameter λ is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

The distribution function is

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

For this distribution, we have

$$E(X) = \frac{1}{\lambda}, \; Var(X) = \frac{1}{\lambda^2}$$

An important property of an exponential random variable X with parameter λ is the *memoryless property*. This property states that for all $x \geq 0$ and $t \geq 0$,

$$\begin{aligned} P(X > t + x | X > t) &= \frac{P(X > t + x, X > t)}{P(X > t)} \\ &= \frac{P(X > t + x)}{P(X > t)} \\ &= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}} \quad = e^{-\lambda x} \quad = P(X > x) \end{aligned}$$

Exponential Distribution, Exp(λ) (2/2)

The ending probability of an exponentially distributed interval

Assume that a call with Exp(λ) distributed duration has lasted the time t. What is the probability that it will end in an infinitesimal interval of length h?

$$\begin{aligned} P(X \le t + h | X > t) &= P(X \le h) \quad \text{(memoryless)} \\ &= 1 - e^{-\lambda h} \\ &= 1 - (1 - \lambda h + \frac{1}{2}(\lambda h)^2 - \cdots) \\ &= \lambda h + o(h) \end{aligned}$$

$\boxed{\text{The ending probability per time unit} = \lambda}$ (constant!)

Geometric Distribution

$$P(X=k)=pq^{k-1}, \quad k=1,2,\cdots$$

$$\begin{aligned}
P(X=j+k|X>j) &= \frac{P(X=j+k)}{P(X>j)} \\
&= \frac{pq^{j+k-1}}{\sum_{i=j+1}^{\infty} pq^{i-1}} \\
&= pq^{k-1}, \quad k=1,2,\cdots \\
&= P(X=k) \quad \text{not a function of } j
\end{aligned}$$

Erlang Distribution, Erlang(n, λ) (1/2)

X is the sum of n independent random variables with the distribution $\mathrm{Exp}(\lambda)$.

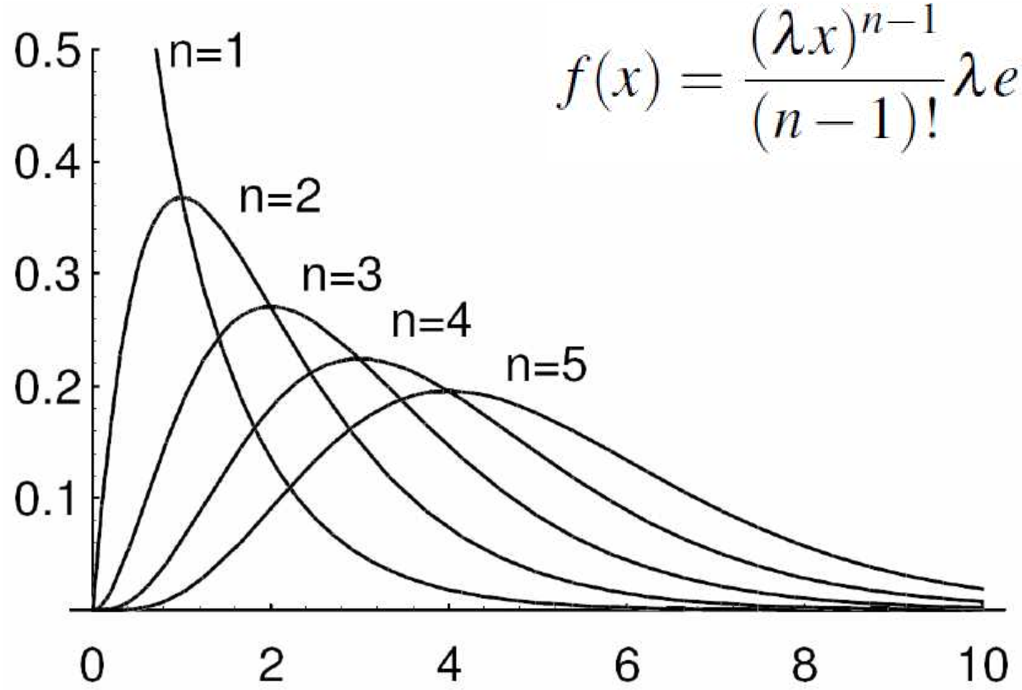
$$X = X_1 + X_2 + \cdots + X_n \quad X_i \sim Exp(\lambda)$$

The Laplace transform is

$$f^*(s) = \left(\frac{\lambda}{\lambda + s}\right)^n$$

By inverse transform (or by recursively convoluting the density function) one obtains the pdf of the sum X

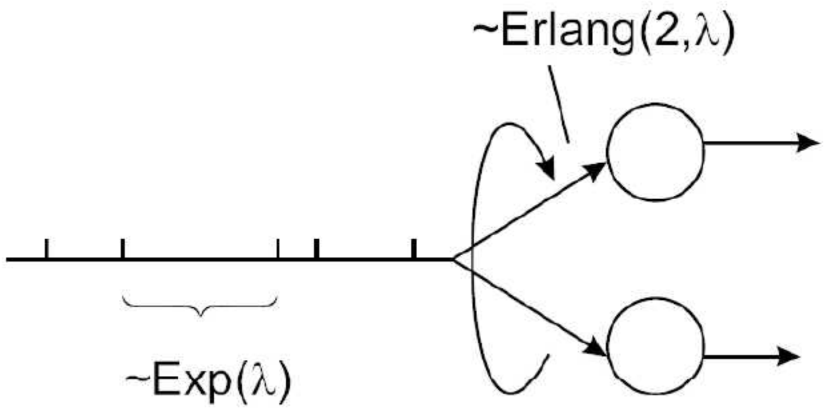
$$f(x) = \frac{(\lambda x)^{n-1}}{(n-1)!}\lambda e^{-\lambda x} \quad x \geq 0$$



Erlang Distribution, Erlang(n, λ) (2/2)

Problem The system consists of two servers. Customers arrive with Exp(λ) distributed interarrival times. Customers are alternately sent to servers 1 and 2. Determine the interarrival time distribution of customers arriving at a given server. □

Solution The interarrival time distribution of customers arriving at a given server is Erlang($2, \lambda$). If $\lambda = 1$, the average interarrival time of customers is 1 second and the average interarrival time of customers at a server is $E(X) = 2/\lambda = 2$ second. □



Gamma Distribution, Gamma(p, λ)

The formula for the pdf of the Erlang distribution can be generalized, from the integer parameter n, to arbitrary real numbers by replacing the factorial $(n-1)!$ by the gamma function $\Gamma(n)$:

$$f(x) = \frac{(\lambda x)^{p-1}}{\Gamma(p)} \lambda e^{-\lambda x}$$

where Gamma function, $\Gamma(p)$, is defined by

$$\Gamma(p) = \int_0^\infty e^t t^{p-1} dt$$

3. Poisson Process (1/3)

- General

A *Poisson process* is one of the simplest interesting stochastic processes.

Definition The stochastic process is a Poisson process with rate $\lambda > 0$ if

1. P(a single event in an interval of duration Δt) = $\lambda\Delta t + o(\Delta t)$
2. P(more than one event in an interval of duration Δt) = $o(\Delta t)$
3. The numbers of events occurring in nonoverlapping intervals of time are independent of each other

where $o(\Delta t)$, $(\Delta t \to 0)$, is a shorthand notation for a function, $f(\Delta t)$ say, for which $\lim_{\Delta t \to 0} f(\Delta t)/\Delta t = 0$. Hence, for small Δt,

$$P(\text{arrival in}(t, t+\Delta t]) \approx \lambda\Delta t.$$

Result Let $P_k(t)$ be the probability that exactly k events occur in an interval of length t, namely, $P_k(t) = P(N(t) = k)$. We have, for each $k \in \mathbb{N}$, $t \geq 0$,

$$P_k(t) = \frac{(\lambda t)^k}{k!}e^{-\lambda t}.$$

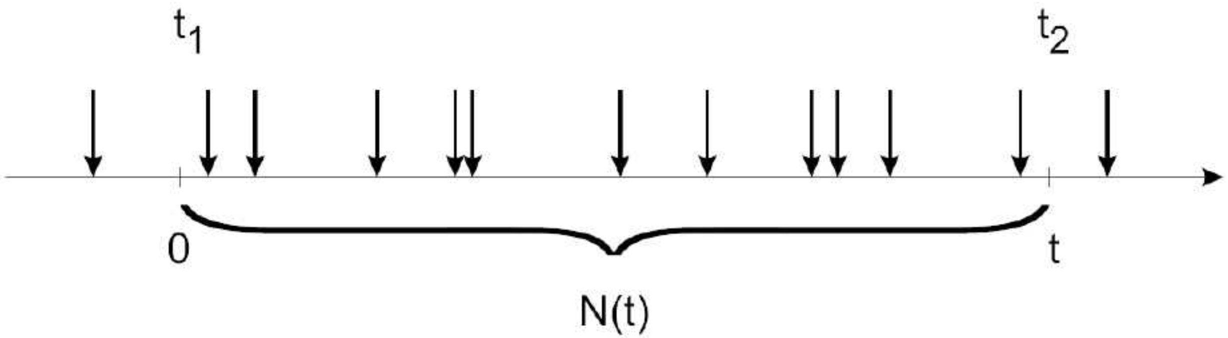
The mean and variance are equal to

$$E(N(t)) = \lambda t, \ \ Var(N(t)) = \lambda t$$

Poisson Process (2/3)

- Counting process

With a slight abuse of notation, $N(t)$, the number of events of a Poisson process in an interval of length t, is sometimes called a Poisson process. Processes like the (continuous-time, discrete-space) stochastic process $(N(t), t \geq 0)$ (or simply $N(t)$) are called *counting processes*.



The counter tells the number of arrivals that have occurred in the interval $(0,t)$ or, more generally, in the interval (t_1,t_2).

$$\begin{cases} N(t) = \text{number of arrivals in the interval } (0,t) \text{ (the stochastic process we consider)} \\ N(t_1,t_2) = \text{number of arrival in the interval} (t_1,t_2) \text{(the increment process } N(t_2) - N(t_1)) \end{cases}$$

Poisson Process (3/3)

- Connection between Poisson process and the exponential distribution

Result Consider the time τ that elapses between the occurrence of two consecutive events in a Poisson process. For each $x \geq 0$

$$P(\tau \leq x) = 1 - e^{-\lambda x}.$$

□

Proof. The proof is very simple. We have

$$\begin{aligned} P(\tau > x) &= P(\text{no event in an interval of length } x) \\ &= e^{-\lambda x} \end{aligned}$$

from Result Therefore, $P(\tau \leq x) = 1 - e^{\lambda x}$ for $x \geq 0$. ■

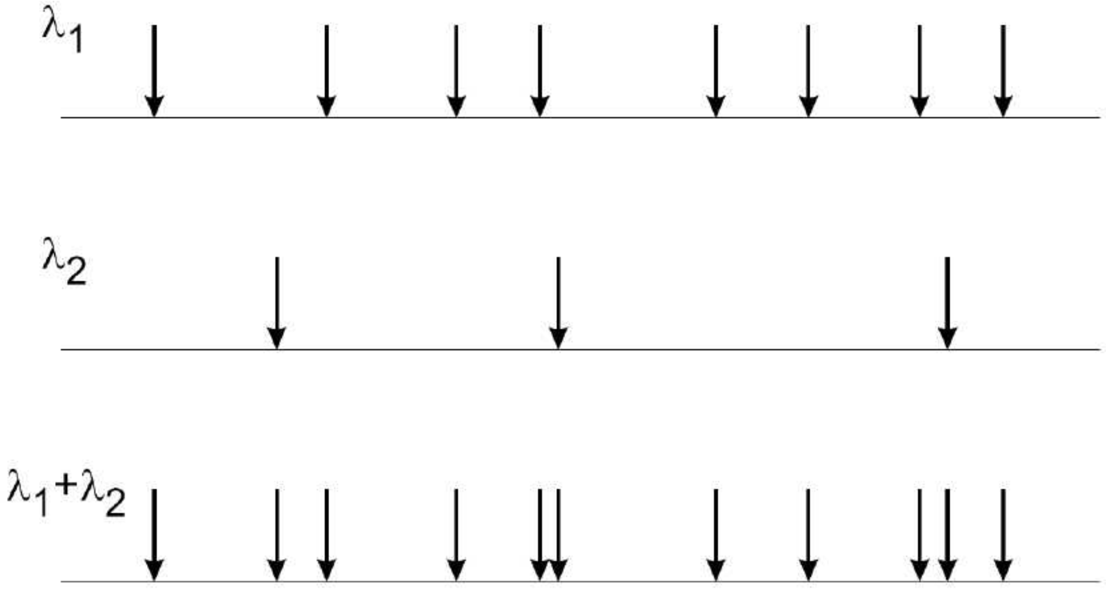
In summary, the sequence $(\tau_n, n = 1, 2, \cdots)$ of interevent times of a Poisson process with rate λ is a sequence of mutually independent r.v.'s, each being exponentially distributed with parameter λ.

Properties of the Poisson Process (1/2)

Superposition (Merging)

Suppose that $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with respective rates λ_1 and λ_2. Then the sum $N_1(t)+N_2(t)$ of the two processes is again a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.

Namely, the superposition of two independent Poisson processes with rates λ_1 and λ_2 is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.



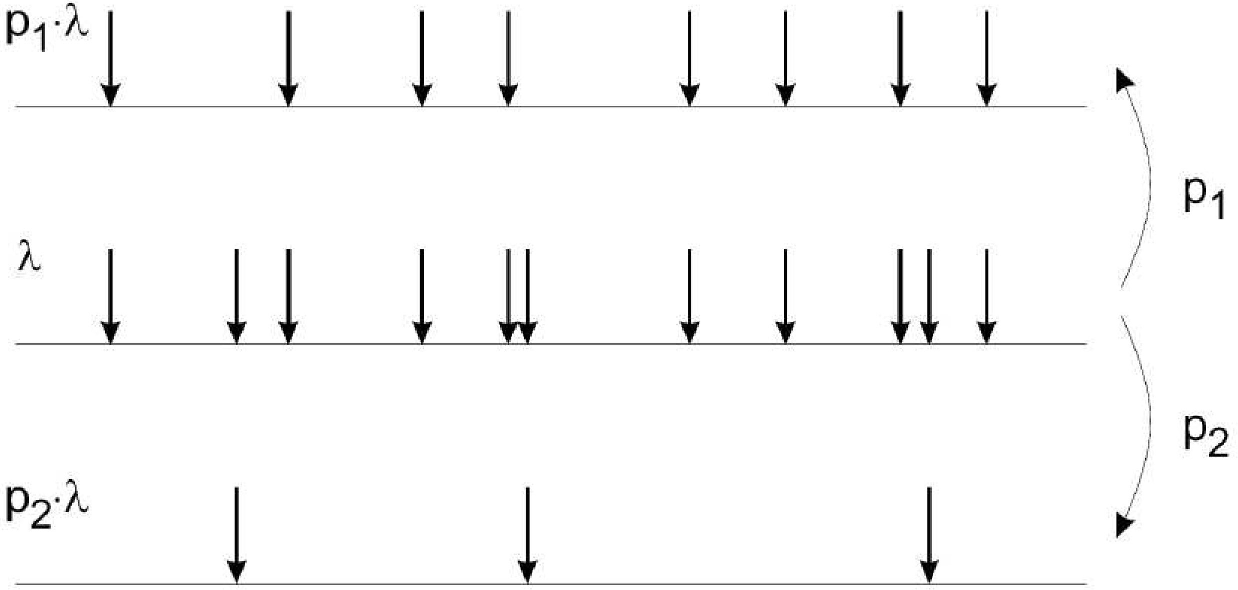
Properties of the Poisson Process (2/2)

Random selection

If a random selection is made from a Poisson process with intensity λ such that each arrival is selected with probability p, independently of the others, the resulting process is a Poisson process with intensity $p\lambda$.

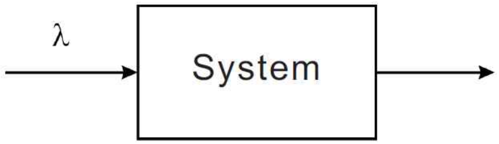
Random splitting

Suppose that $A(t)$ is a Poisson process with rate λ and that each arrival is marked with probability p independent of all other arrivals. Let $N_1(t)$ and $N_2(t)$ denote respectively the number of marked and unmarked arrivals in $[0,t]$. Then $N_1(t)$ and $N_2(t)$ are both Poisson processes with respective rates $p\lambda$ and $(1-p)\lambda$. And these two processes are independent.



Poisson Arrivals See Time Average (PASTA)

Consider an arbitrary system which spends its time in different states E_j. Arrivals to the system constitute a Poisson process with intensity λ. These arrivals induce state transitions in the system.



In equilibrium, we may associate with each state E_j two different probabilities:

- The probability of the state as seen by an outside random observer

$$\begin{aligned}\pi_j &= \text{probability that the system is in the state } E_j \text{ at a random instant} \\ &= P(\text{there are } E_j \text{ states in system}) \\ &= \lim_{T\to\infty} \frac{\text{total time that there are } E_j \text{ states in system in } [0,T]}{T}\end{aligned}$$

- The probability of the state seen by an arriving customer

$$\begin{aligned}\pi_j^* &= \text{probability that the system is in the state } E_j \text{ just before (a randomly chosen) arrival} \\ &= P(\text{an arrival sees } E_j \text{ state in system})\end{aligned}$$

In the case of a Poisson arrival process it holds

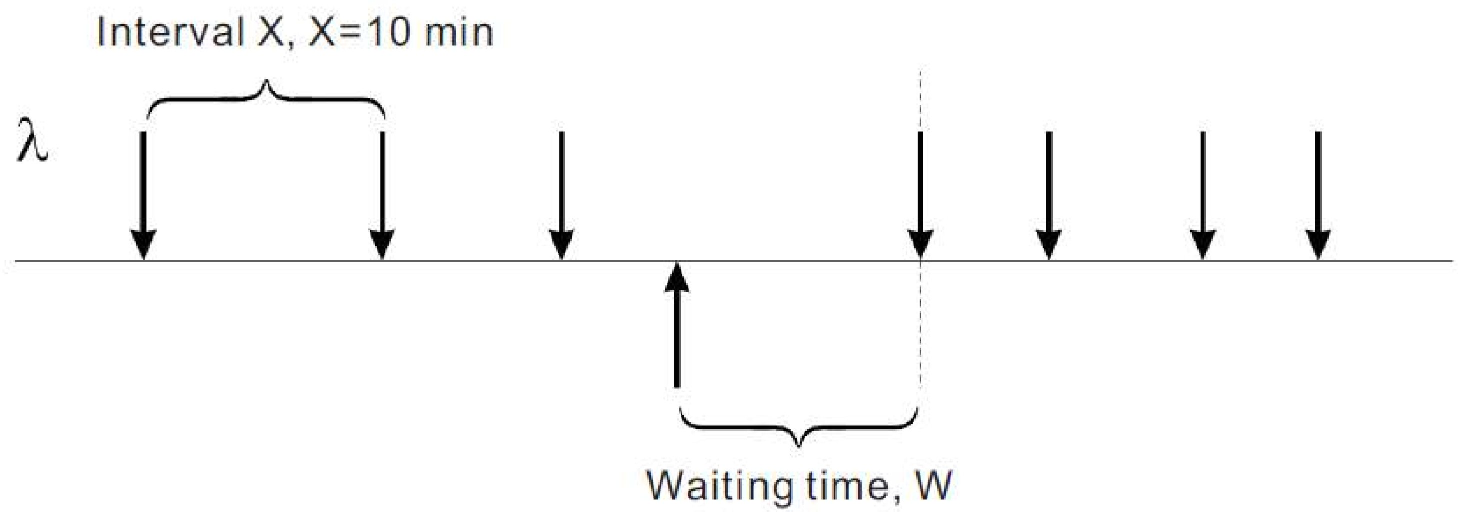
$$\pi_j = \pi_j^*$$

The Hitchhiker's Paradox (1/2)

The setting of the paradox is the following

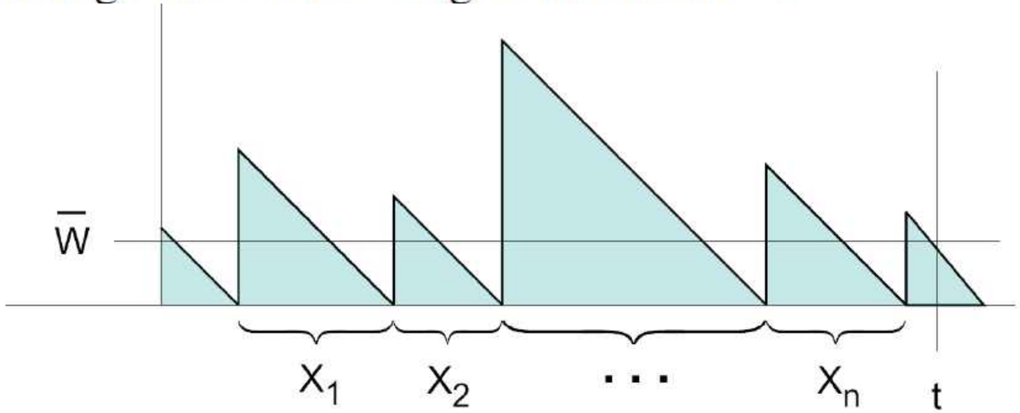
- Cars are passing a point of a road according to a Poisson process.
- The mean interval between the cars is 10 min.
- A hitchhiker arrives to the roadside point at random instant of time.
- What is the mean waiting time $\overline{W}$ until the next car.

The interarrival times in a Poisson process are exponentially distributed. From the memoryless property of the exponential distribution it follows that the (residual) time to the next arrival has the same Exp(λ) distribution and the expected time is thus $\overline{W} = 10$ min. This appears paradoxical. Why isn't the expected time 5 min? Is there something wrong? Answer: No, the expected time is indeed $\overline{W} = 10$ min.



The Hitchhiker's Paradox (2/2)

Consider a long period of time t. The waiting time to the next car arrival $W(\tau)$ as the function of the arrival instant of the hitchhiker τ is represented by the sawtooth curve in the figure. The mean waiting time is the average value of the curve.



$$\overline{W} = \frac{1}{t}\int_0^t W(\tau)d\tau \approx \frac{1}{t}\sum_{i=1}^{n}\frac{1}{2}X_i^2 \quad \text{(sum of the ares of the triangles)}$$

where X_i is the interarrival time. As $t \to \infty$ the number of the triangles n tends to $t/\overline{X}$.

$$\overline{W} = \frac{1}{\overline{X}}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}X_i^2 = \frac{1}{2}\frac{\overline{X^2}}{\overline{X}}$$

For exponential distribution, $\overline{X^2} = (\overline{X})^2 + \underbrace{V(X)}_{(\overline{X})^2} = 2(\overline{X})^2$

thus $\overline{W} = \overline{X}$.

Summary

- Order Statistics
  - When $X_{(n)}$ denotes the nth smallest of element of the sample X. This statistics is called the order statistic of order n.

- Common Distributions
  - Exponential Distribution: memoryless property
  - Geometric Distribution
  - Erlang Distribution

- Poisson Process
  - Poisson process vs. exponential distribution
  - Properties: superposition, Random selection, Random splitting
  - PASTA