대기행렬 이론 개론

III. Markov Process

학습에 앞서

- 학습 목표
  - 마코프 프로세스 개념을 이해한다.
  - 이산시간 마코프 체인과 연속시간 마코프 체인을 학습한다.
  - 상태전이확률 및 정상상태확률을 구하는 능력을 배양한다.

- 목차
  - 1. Markov Process
  - 2. Discrete-Time Markov Chain (DTMC)
  - 3. Continuous-Time Markov Chain (CTMC)

1. Markov Process

Definition A Markov process is a stochastic process $(X(t), t \in T)$, $X(t) \in E \subset R$, such that

$$P(X(t_{n+1}) \leq x_{n+1} | X(t_1) = x_1, \cdots, X(t_n) = x_n)$$
$$= P(X(t_{n+1}) \leq x_{n+1} | X(t_n) = x_n)$$

for all $x_1, \cdots, x_{n+1} \in E$, $t_1, \cdots, t_{n+1} \in T$ with $t_1 < t_2 < \cdots < t_{n+1}$. □

Table Classifications of Markov processes

T (time index) I (state-space)	countable	uncountable
countable	discrete-space discrete-time Markov process	discrete-space continuous-time Markov process
uncountable	continuous-space discrete-time Markov process	continuous-space continuous-time Markov process

2. Discrete-Time Markov Chain (DTMC)

- Transition Probability

A discrete-time Markov Chain (DTMC) is a discrete-time (with index set $\mathbb{N}$) discrete-space (with state-space $I = \mathbb{N}$ if infinite and $I \subset \mathbb{N}$ if finite) stochastic process $(X_n, n \in \mathbb{N})$ such that for all $n \geq 0$

$$P(X_{n+1} = j | X_0 = i_0, X_1 = i_1, \cdots, X_{n-1} = i_{n-1}, X_n = i) = P(X_{n+1} = j | X_n = i)$$

for all $i_0, \cdots, i_{n-1}, i, j, \in I$. A DTMC. is called a *finite-state* Markov chain if the set I is finite.

A DTMC is *homogeneous* if $P(X_{n+1} = j | X_n = i)$ does not depend on n for all $i, j \in I$. If so, we shall write

$$p_{ij} := P(X_{n+1} = j | X_n = i) \quad \forall i, j \in I,$$

where p_{ij} is the *one-step transition probability* from state i to state j. Unless otherwise mentioned we shall only consider homogeneous DTMC's.

The probability of a path $i_0, i_1, \cdots, i_n$ is

$$P(X_0 = i_0, \cdots, X_n = i_n) = P(X_0 = i_0) p_{i_0,i_1} p_{i_1,i_2} \cdots p_{i_{n-1},i_n}$$

Transition Matrix

- Transition Matrix

Define $\mathbf{P}$ be the *transition matrix* of a DTMC, namely,

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0j} & \cdots \\ p_{10} & p_{11} & \cdots & p_{1j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i0} & p_{i1} & \cdots & p_{ij} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$p_{ij} \geq 0, \quad \forall i, j \in I$$

$$\sum_{j \in I} p_{ij} = 1, \quad \forall i \in I.$$

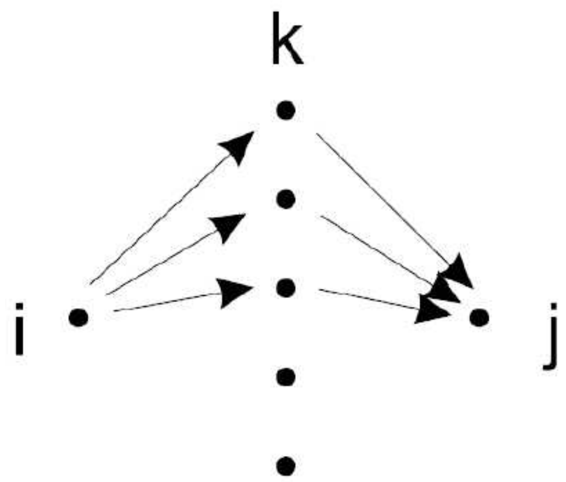
Chapman-Kolmogorov Equation

We now define the n-step transition probabilities $p_{ij}^{(n)}$ by

$$p_{ij}^{(n)} = P(X_n = j | X_0 = i)$$

Result Chapman-Kolmogorov equation
For all $n \geq 0$, $m \geq 0$, $i, j \in I$, we have

$$p_{ij}^{(n+m)} = \sum_{k \in I} p_{ik}^{(n)} p_{kj}^{(m)}$$



or, in matrix notation,

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \mathbf{P}^{(m)}$$

where $\mathbf{P}^{(n)} := \left[p_{ij}^{(n)} \right]$. Therefore,

$$\mathbf{P}^{(n)} = \mathbf{P}^{n} \quad \forall n \geq 1$$

where P^n is the n-th power of the matrix $\mathbf{P}$.

Problem

Problem Consider a communication system that transmits the digits 0 and 1 through several stages. At each stage, the probability that the same digit will be received by the next stage is 0.75. What is the probability that a 0 that is entered at the first stage is received as a 0 at the fifth stage? □

Solution We want to find $p_{00}^{(5)}$ for a DTMC with transition matrix $\mathbf{P}$ given by

$$\mathbf{P} = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}$$

We know that $p_{00}^{(5)}$ is the $(1,1)$-entry of the matrix $\mathbf{P}^5$. We find $p_{00}^{(5)} = 0.515625$. □

Steady-state Probability, Limiting Probability (1/3)

we would like to compute $\pi_j^{(n)} := P(X_n = j)$.

Example Consider a DTMC with transition matrix P given by

$$\mathbf{P} = \begin{pmatrix} 1-p & p(1-p) & p^2 \\ 1-p & p(1-p) & p^2 \\ 0 & 1-p & p \end{pmatrix} \quad p = 1/3$$

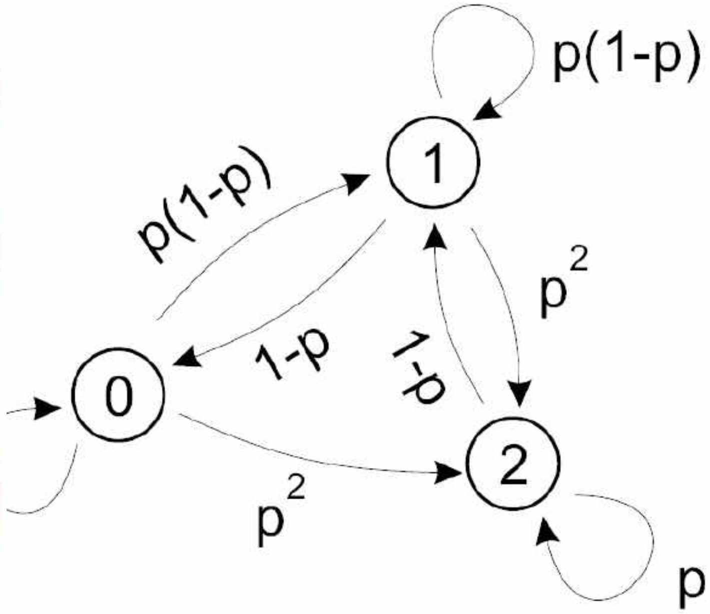
$$\mathbf{P} = \frac{1}{9}\begin{pmatrix} 6 & 2 & 1 \\ 6 & 2 & 1 \\ 0 & 6 & 3 \end{pmatrix} = \begin{pmatrix} 0.6666 & 0.2222 & 0.1111 \\ 0.6666 & 0.2222 & 0.1111 \\ 0 & 0.6666 & 0.3333 \end{pmatrix}$$

$$\mathbf{P}^2 = \frac{1}{9^2}\begin{pmatrix} 48 & 22 & 11 \\ 48 & 22 & 11 \\ 36 & 30 & 15 \end{pmatrix} = \begin{pmatrix} 0.5926 & 0.2716 & 0.1358 \\ 0.5926 & 0.2716 & 0.1358 \\ 0.4444 & 0.3704 & 0.1852 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{P}^3 = \frac{1}{9^3}\begin{pmatrix} 420 & 206 & 103 \\ 420 & 206 & 103 \\ 396 & 222 & 111 \end{pmatrix} = \begin{pmatrix} 0.5761 & 0.2826 & 0.1413 \\ 0.5761 & 0.2826 & 0.1413 \\ 0.5432 & 0.3045 & 0.1523 \end{pmatrix}$$

$$\mathbf{P}^8 = \frac{1}{9^2}\begin{pmatrix} 0.5714 & 0.2857 & 0.1429 \\ 0.5714 & 0.2857 & 0.1429 \\ 0.5714 & 0.2857 & 0.1429 \end{pmatrix}$$



Steady-state Probability, Limiting Probability (2/3)

From the above example, regardless of the initial state, the Markov process reaches a steady state as follows:

$$\pi_j = \lim_{n \to \infty} p_{ij}^{(n)},$$

which is called limiting probability or steady state probability. In steady state, the probability that a state i appears does not change with time.

Result Meaning of π_j is

- Ensemble-average probability: π_j is the probability that the state is j when an observer sees the Markov chain at $n \to \infty$.
- Time-average probability: π_j is the time ratio that the state j stays.

Steady-state Probability, Limiting Probability (2/3)

Result Let $\pi_j := P(X_n = j)$ in steady state. If a DTMC with transition matrix $\mathbf{P}$ is irreducible and aperiodic, we can compute π_j by solving equations as follows:

$$\pi_j = \sum_{i \in I} \pi_i p_{ij}$$

$$\sum_{j \in I} \pi_j = 1$$

or

$$\pi = \pi \mathbf{P} \quad \text{Balance equation}$$

$$\pi \mathbf{1} = 1 \quad \text{Normalization equation.}$$

where $\mathbf{1} = [1 \ \cdots \ 1]^T$.

Remarks on the Stationary Distribution

If the limit probabilities (the components of the vector) π exist, they must satisfy the equation $\pi = \pi\mathbf{P}$, because

$$\pi = \lim_{n\to\infty} \pi^{(n)} = \lim_{n\to\infty} \pi^{(n+1)} = \lim_{n\to\infty} \pi^{(n)} \cdot \mathbf{P} = \pi\mathbf{P}$$

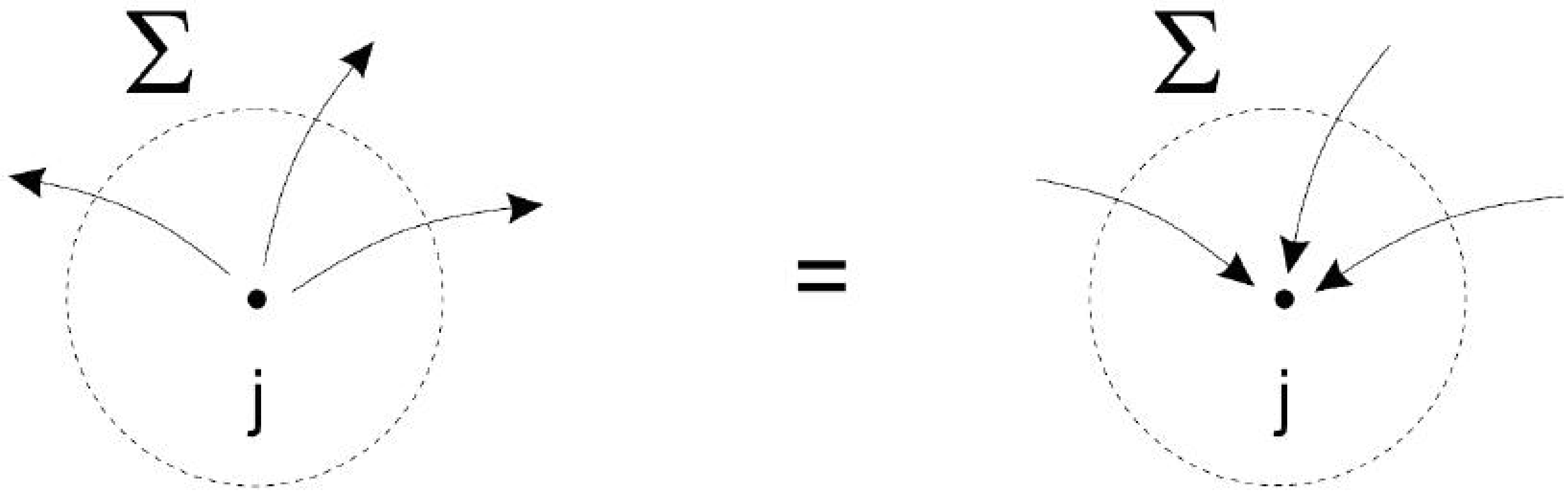
The equation $\pi = \pi\mathbf{P}$ can also be expressed in the form: π is the (left) eigenvector of the matrix $\mathbf{P}$ belonging to the eigenvalue 1 (or belonging to the eigenvalue 0 of the matrix $(\mathbf{P}-\mathbf{I})$).

The limit distribution π is called the *stationary distribution* or the *equilibrium distribution*.

Global Balance

Equation $\pi = \pi\mathbf{P}$ or $\pi_j = \sum_i \pi_i p_{i,j}$, $\forall j$, is often called the (global) balance condition. Since $\sum_i p_{j,i} = 1$ (the transition takes the system to some state), one can write

$$\underbrace{\sum_i \pi_j p_{j,i}}_{\text{prob. that the system is in state } j \text{ and makes a transition to another state}} = \underbrace{\sum_i \pi_i p_{i,j}}_{\text{prob. that the system is in another state and makes a transition to state } j} = \underbrace{\text{One equation for each state } j.}_{\text{Balance of probability flows: there are as many exits from state } j \text{ as there are entries to it.}}$$



3. Continuous-Time Markov Chain (CTMC)

- Transition Probability

A continuous-time Markov chain (denoted as CTMC) is a continuous-time (with index set $[0,1)$), discrete-space (with state-space I) stochastic process $(X(t), t > 0)$ such that

$$P(X(t_{n+1}) = j | X(t_0) = i_0, X(t_1) = i_1, \cdots, X(t_{n-1}) = i_{n-1}, X(t_n) = i)$$
$$= P(X(t_{n+1}) = j | X(t_n) = i)$$

for all $i_0, \cdots, i_{n-1}, i, j \in I, 0 \leq t_0 < \cdots < t_n < t_{n+1}$.

Due to the Markov property, the time the system spends in any given state is memoryless: the distribution of the remaining time depends solely on the state but not on the time already spent in the state) the time is exponentially distributed.

A CTMC is homogeneous if

$$p_{ij}(t) := P(X(t) = j | X(0) = i) = P(X(t+s) = j | X(s) = i)$$

for all $i, j \in I, 0 \leq s < t, u \geq 0$.

Rate Matrix

- Rate Matrix

Define
- q_{ii}: the rate at which it departs state i
- q_{ij}: the rate at which it moves from state i to state j, $j \neq i$

$$q_{ii} := \lim_{\Delta t \to 0} \frac{p_{ii}(\Delta t) - 1}{\Delta t} \leq 0 \qquad q_{ij} := \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta t} \geq 0,$$

The total transition rate out of state i is $q_i = \sum_{j \neq i} q_{ij}$

Define $q_{ii} = -q_i$

$$\mathbf{Q} = \begin{pmatrix} -\sum_{j\neq 0} q_{0j} & q_{01} & \cdots & q_{0j} & \cdots \\ q_{10} & -\sum_{j\neq 1} q_{1j} & \cdots & q_{1j} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{i0} & q_{i1} & \cdots & -\sum_{j\neq i} q_{ij} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \qquad \sum_{j \in I} q_{ij} = 0 \quad \forall i \in I.$$

Chapman-Komogorov Equation

Result (Chapman-Kolmogrov equation)
For all $t > 0, s > 0, i, j \in I$, we have

$$p_{ij}(t+s) = \sum_{k \in I} p_{ik}(t) p_{kj}(s).$$

Steady-state Probability, Limiting Probability

Result Let $\pi_j := P(X(t) = j)$ in steady state. If a CTMC with infinitesimal generator $\mathbf{Q}$ is irreducible, we can compute $\pi = [\pi_0, \pi_1, \cdots]$ by solving equations as follows:

$$\pi\mathbf{Q} = 0$$

$$\pi\mathbf{1} = 1 \quad \text{Normalization equation.}$$

$$\sum_{j \in I} \pi_j q_{ji} = 0$$

$$\sum_{i \neq j} \pi_j q_{j,i} = \sum_{i \neq j} \pi_i q_{i,j} \quad \text{Balance equation.}$$

the probability flow out of a state = the probability flow into that state

Global Balance

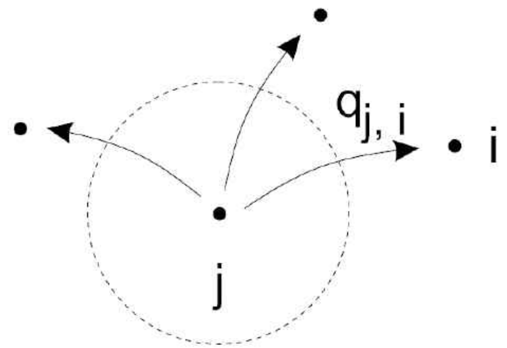
The stationary solution $\pi = \lim_{t \to \infty} \pi(t)$ is independent of time and thus satisfies

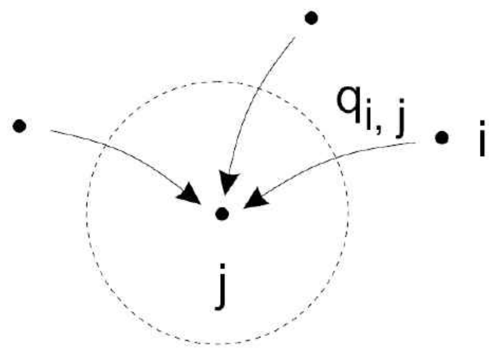
$$\pi \cdot \mathbf{Q} = \mathbf{0}$$

Global balance condition which expresses the balance of probability flows.
The jth row is

$$\sum_{i \neq j} \pi_j q_{j,i} = \sum_{i \neq j} \pi_i q_{i,j}$$

where $\pi_i q_{i,j}$ is the probability flow from state i to state j





Summary

- Markov Process
  - The future state is independent of its past states conditionally on the present.

- Discrete-Time Markov Chain (DTMC)
  - Transition Probability, p_{ij}
  - Transition Matrix, **P**
  - Steady-state Probability
  - Balance Equation

$$\pi_j = \sum_{i \in I} \pi_i p_{ij}$$

$$\sum_{j \in I} \pi_j = 1$$

- Continuous-Time Markov Chain (CTMC)
  - Transition Probability
  - Balance Equation

$$\sum_{j \in I} \pi_j q_{ji} = 0$$

$$\sum_{i \neq j} \pi_j q_{j,i} = \sum_{i \neq j} \pi_i q_{i,j} \quad \text{Balance equation.}$$