대기행렬 이론 개론

VI. M/M/c Queue

학습에 앞서

- 학습 목표
  - M/M/c, M/Mc/K, M/M/c/c, M/M/∞ 대기행렬 모델을 이해한다.
  - 상태전이확률을 구하고 정상상태확률을 구하는 능력을 배양한다.

- 목차
  - 1. The M/M/c Queue
  - 2. The M/M/c/K Queue
  - 3. The M/M/c/c Queue (Erlang loss system)
  - 4. The M/M/∞ Queue

1. The *M/M/c* Queue

- Introduction

There are $c \geq 1$ servers and the waiting room has infinite capacity. If more than one server is available when a new customer arrives (which necessarily implies that the waiting room is empty) then the incoming customer may enter any of the free servers.

Let λ and μ be the rate of the Poisson process for the arrivals and the parameter of the exponential distribution for the service times, respectively.

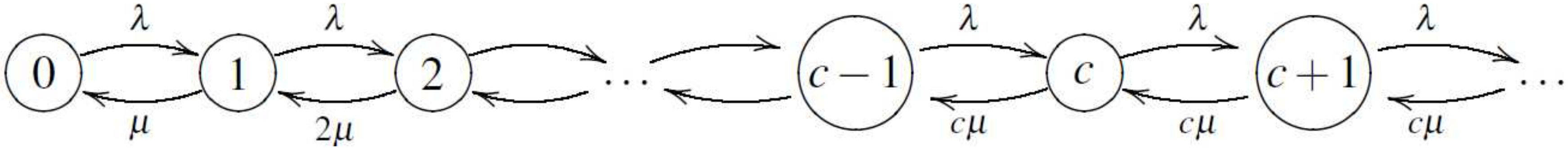
State Transition Rate Diagram

Here again the process $(X(t), t \geq 0)$ of the number of customers in the system can be modeled as a birth and death process. The birth rate and the death rate are

$$\lambda_i = \lambda, \qquad i \geq 0$$

$$\mu_i = \begin{cases} i\mu & , \text{if } i = 1, 2, \cdots, c-1 \\ c\mu & , \text{if } i \geq c \end{cases}$$

which can be also written as $\mu_i = \mu \min(i, c)$ for all $i \geq 1$.



State transition rate diagram for $M/M/c$

The process of $M/M/1$ is a birth-death process with birth rate $\lambda_i = \lambda$ and with death rate $\mu_i = \mu$.

Steady-state Probability

- Birth-death process

$$\pi_i = \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^{i} \mu_k} \pi_0$$

M/M/c queue

$$\pi_i = \begin{cases} \frac{a^i}{i!}\pi_0 & \text{, if } 1 \le i \le c-1 \\ \frac{a^c}{c!}\rho^{i-c}\pi_0 & \text{, if } i \ge c \end{cases}$$

where the traffic intensity is $\rho = \frac{\lambda}{c\mu}$, the offered load is $a = \lambda E(S) = \frac{\lambda}{\mu}$, and

$$\pi_0 = \left[\sum_{i=0}^{c-1} \frac{a^i}{i!} + \frac{a^c}{c!} \left(\frac{1}{1-\rho} \right) \right]^{-1}$$

Performance Measure (1/2)

Result (Erlang's C formula) The probability that an arriving customer is forced to join the queue to wait for service is given by

$$P_{\text{wait}} = \sum_{i=c}^{\infty} \pi_i = \sum_{i=c}^{\infty} \frac{a^c}{c!} (a/c)^{i-c} \pi_0$$

$$= \frac{a^c}{c!(1-a/c)} \pi_0 = \frac{\dfrac{a^c}{c!(1-a/c)}}{\sum_{i=0}^{c-1} \dfrac{a^i}{i!} + \dfrac{a^c}{c!(1-a/c)}}, \quad a = \lambda E(S) < c$$

This probability is of wide use in telephony and gives the probability that no trunk (i.e., server) is available for an arriving call (i.e., customer) in a system of c trunks. It is referred to as Erlang's C formula as follows:

$$C(c,a) = P_{\text{wait}} = \frac{a^c}{c!(1-a/c)} \pi_0$$

Performance Measure (2/2)

- Average number of waiting customers in the queue:

$$L_q = \sum_{i=c}^{\infty}(i-c)\pi_i = \frac{(c\rho)^c}{c!} \cdot \frac{\rho}{(1-\rho)^2}\pi_0 = \frac{\rho}{(1-\rho)^2}\pi_c.$$

- Average number of customers in the system:

$$L = L_q + a' = L_q + \frac{\lambda}{\mu}.$$

- Average sojourn time in the system:

$$W = \frac{L}{\lambda} = W_q + \frac{1}{\mu} \quad \text{Little's formula.}$$

- Average waiting time in the queue:

$$W_q = \frac{L_q}{\lambda} \quad \text{Little's formula.}$$

2. The *M/M/c/K* Queue

The number of customers is limited less than or equal to K. We consider $K \geq c$.

The birth rate and the death rate are

$$\lambda_i = \begin{cases} \lambda & \text{, if } 0 \leq i \leq K-1 \\ 0 & \text{, if } i \geq K. \end{cases} \qquad \mu_i = \begin{cases} i\mu & \text{, if } 1 \leq i \leq c \\ c\mu & \text{, if } c+1 \leq i \leq K \end{cases}$$

Steady-state Probability

The steady-state probability is

$$\pi_0 = \left[\sum_{j=0}^{c-1} \frac{a^j}{j!} + \frac{a^c}{c!}\left(\frac{1-\rho^{K-c+1}}{1-\rho}\right)\right]^{-1}, \quad \rho = \frac{\lambda}{c\mu}, a = \lambda E(S) = \frac{\lambda}{\mu}$$

$$\pi_i = \begin{cases} \frac{a^i}{i!}\pi_0 & \text{, if } 0 \le i \le c \\ \frac{a^c}{c!}\rho^{i-c}\pi_0 & \text{, if } c+1 \le i \le K \end{cases}$$

Birth-death process

$$\pi_i = \frac{\prod_{k=0}^{i-1}\lambda_k}{\prod_{k=1}^{i}\mu_k}\pi_0$$

M/M/c queue

$$\pi_i = \begin{cases} \frac{a^i}{i!}\pi_0 & \text{, if } 1 \le i \le c-1 \\ \frac{a^c}{c!}\rho^{i-c}\pi_0 & \text{, if } i \ge c \end{cases}$$

Performance Measure

- Average number of waiting customers in the queue:

$$L_q = \sum_{i=c}^{K}(i-c)\pi_i = \frac{a^c}{c!} \cdot \frac{\rho \pi_0}{(1-\rho^2} \cdot \left[1 - \left\{(K-c)(1-\rho)+1\right\}\rho^{K-c}\right]$$

- Average number of customers in the system:

$$L = L_q + a' = L_q + \frac{\lambda}{\mu}(1-\pi_K).$$

- Average sojourn time in the system:

$$W_q = \frac{L_q}{\lambda_e} = W_q + \frac{1}{\mu} \quad \text{Little's formula.}$$

- Average waiting time in the queue:

$$W = \frac{L}{\lambda_e} = \frac{L_q}{\lambda(1-\pi_K)} \quad \text{Little's formula.}$$

3. The *M/M/c/c* Queue (Erlang loss system)

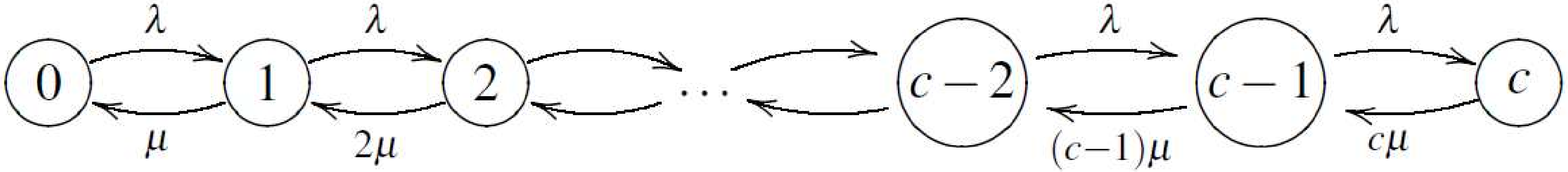
Here we have a situation when there are $c \geq 1$ available servers but no waiting room. This is a pure *loss queueing system*. Each newly arriving customer is given its private server; however, if a customer arrives when all the servers are occupied, that customer is lost.

Let λ and μ be the rate of the Poisson process for the arrivals and the parameter of the exponential distribution for the service times, respectively.

Steady-state Probability

The number of busy servers can be modeled as a birth and death process. The birth rate and the death rate are

$$\lambda_i = \begin{cases} \lambda & , \text{if } i = 0, 1, \cdots, c-1 \\ 0 & , \text{if } i \geq c. \end{cases} \qquad \mu_i = \begin{cases} i\mu & , \text{if } i = 1, 2, \cdots, c-1 \\ c\mu & , \text{if } i \geq c \end{cases}$$



State transition rate diagram for $M/M/c/c$

The steady-state probability is

$$\pi_i = \frac{a^i}{i!}\pi_0, \quad \text{for } 0 \leq i \leq c$$

and $\pi_i = 0$ for $i > c$, where

$$\pi_0 = \left[\sum_{i=0}^{c} \frac{a^i}{i!}\right]^{-1}$$

and $a = \lambda E(S) = \lambda/\mu$.

Performance Measure (1/2)

Result (Erlang's loss formula, Erlang's B formula) This $M/M/c/c$ system is also of great interest in telephony. In particular, π_c gives the probability that all trunks (i.e., servers) are busy, and it is given by

$$\pi_c = \frac{a^c/c!}{\sum_{k=0}^{c} a^k/k!}.$$

This is the celebrated *Erlang's loss formula* (derived by A. K. Erlang in 1917) as follows:

$$B(c,a) = \pi_c = \frac{a^c/c!}{\sum_{k=0}^{c} a^k/k!} = \frac{e^{-a}a^c/c!}{\sum_{k=0}^{c} e^{-a}a^k/k!}$$

$$B(c,\rho) = \frac{\rho^c/c!}{\sum_{k=0}^{c} \rho^k/k!} = \frac{e^{-\rho}\rho^c/c!}{\sum_{k=0}^{c} e^{-\rho}\rho^k/k!}.$$

Remarkably enough Result is valid for any service time distribution and not only for exponential service times! Such a property is called an insensitivity property. Later on, we will see an extremely useful extension of this model to (in particular) several classes of customers, that has nice applications in the modeling and performance evaluation of multimedia networks.

Performance Measure (2/2)

- Average number of waiting customers in the queue:

$$L_q = 0.$$

- Average number of customers in the system:

$$L = L_q + a' = \frac{\lambda}{\mu}(1 - \pi_c).$$

- Average sojourn time in the system:

$$W = \frac{L}{\lambda_e} = W_q + \frac{1}{\mu} = \frac{1}{\mu} \quad \text{Little's formula.}$$

- Average waiting time in the queue:

$$W_q = \frac{L_q}{\lambda_e} = 0 \quad \text{Little's formula.}$$

4. The M/M/∞ Queue

Here we have a situation when an incoming customer is immediately served without waiting, e.g., a park, a public bathroom, etc.

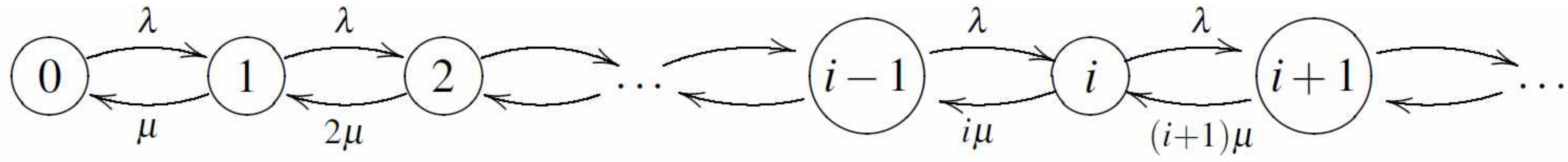
Let λ and μ be the rate of the Poisson process for the arrivals and the parameter of the exponential distribution for the service times, respectively.

State Transition Rate Diagram

The number of customers can be modeled as a birth and death process. The birth rate and the death rate are

$$\lambda_i = \lambda$$

$$\mu_i = i\mu$$



State transition rate diagram for $M/M/\infty$

Steady-state Probability

the steady-state probability is

$$\pi_0 = \left(1 + \sum_{i=1}^{\infty} \frac{\prod_{k=0}^{i-1} \lambda}{\prod_{k=1}^{i} k\mu}\right)^{-1}$$

$$\pi_i = \frac{\prod_{k=0}^{i-1} \lambda}{\prod_{k=1}^{i} k\mu} p_0.$$

- Birth-death process

$$\pi_i = \frac{\prod_{k=0}^{i-1} \lambda_k}{\prod_{k=1}^{i} \mu_k} \pi_0$$

M/M/c queue

$$\pi_i = \begin{cases} \frac{a^i}{i!}\pi_0 & \text{, if } 1 \le i \le c-1 \\ \frac{a^c}{c!}\rho^{i-c}\pi_0 & \text{, if } i \ge c \end{cases}$$

Summary

- Queueing Models
  - The M/M/c Queue
  - The M/M/c/K Queue
  - The M/M/c/c Queue (Erlang loss system)
  - The M/M/∞ Queue
- Performance Measure
  - Average number of customers
  - Average sojourn time
  - Average waiting time

- Eralang's B formula (M/M/c/c)

$$B(c,a) = \pi_c = \frac{a^c/c!}{\sum_{k=0}^{c} a^k/k!} = \frac{e^{-a}a^c/c!}{\sum_{k=0}^{c} e^{-a}a^k/k!}$$

$$B(c,\rho) = \frac{\rho^c/c!}{\sum_{k=0}^{c} \rho^k/k!} = \frac{e^{-\rho}\rho^c/c!}{\sum_{k=0}^{c} e^{-\rho}\rho^k/k!}.$$