신호 및 시스템

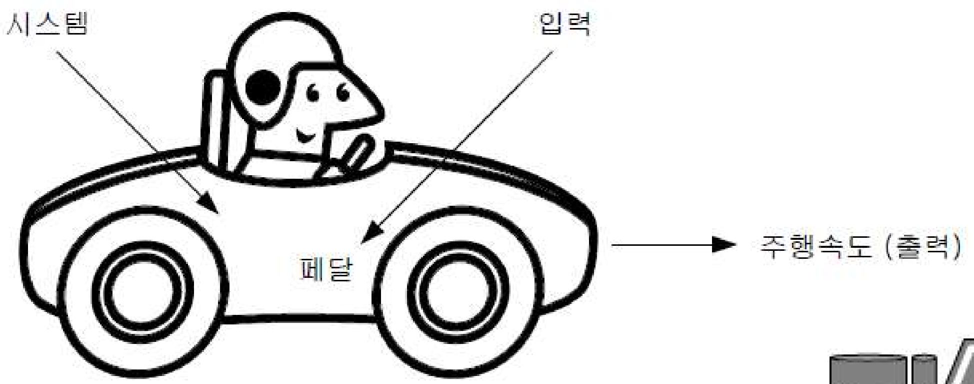
I. 신호의 스펙트럼 표현

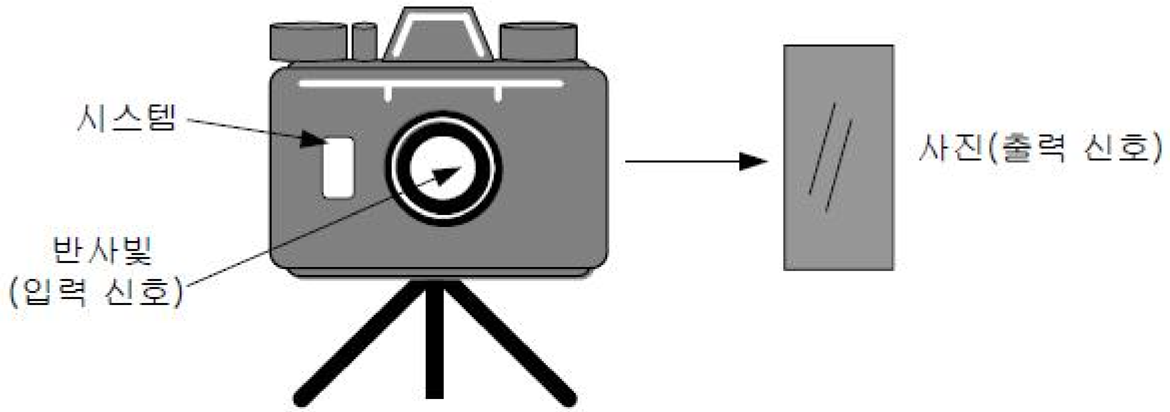
학습에 앞서

- 학습목표
  - 주파수 공간에서 신호의 표현 방법을 학습한다.
  - 신호가 코사인 파에 의해 합성될 수 있음을 확인한다.
  - 푸리에 급수를 이해한다.

Overview (1/2)

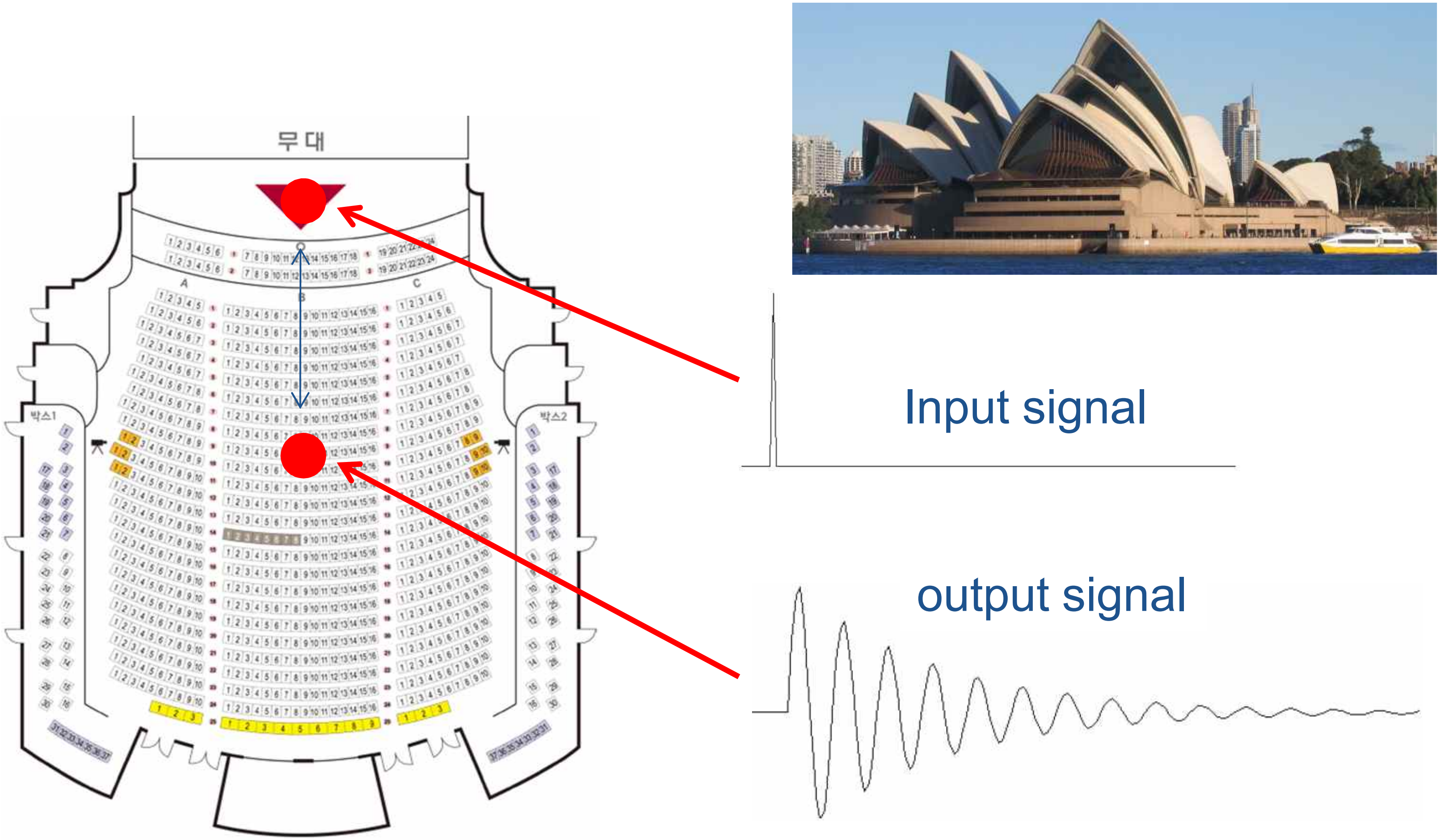
- What do you think of a signal?
  - Something that carries information
- What do you think of a system?
  - Something that can manipulate, change, record, or transmit signals.





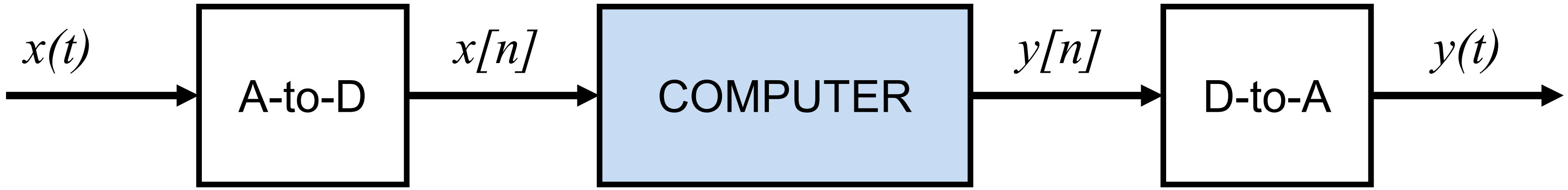
Overview (2/2)

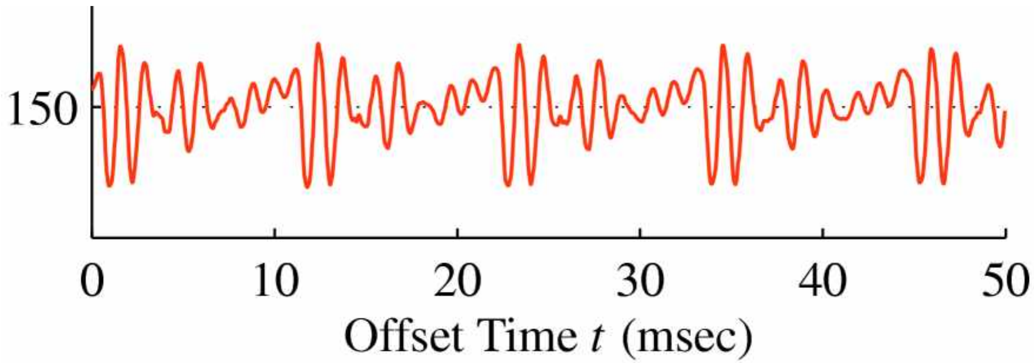
- Opera House

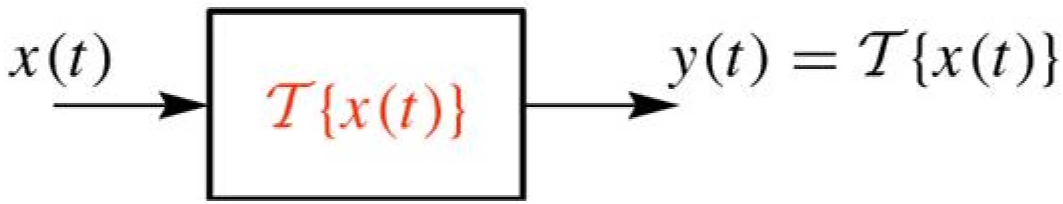


Mathematical Representation

- System

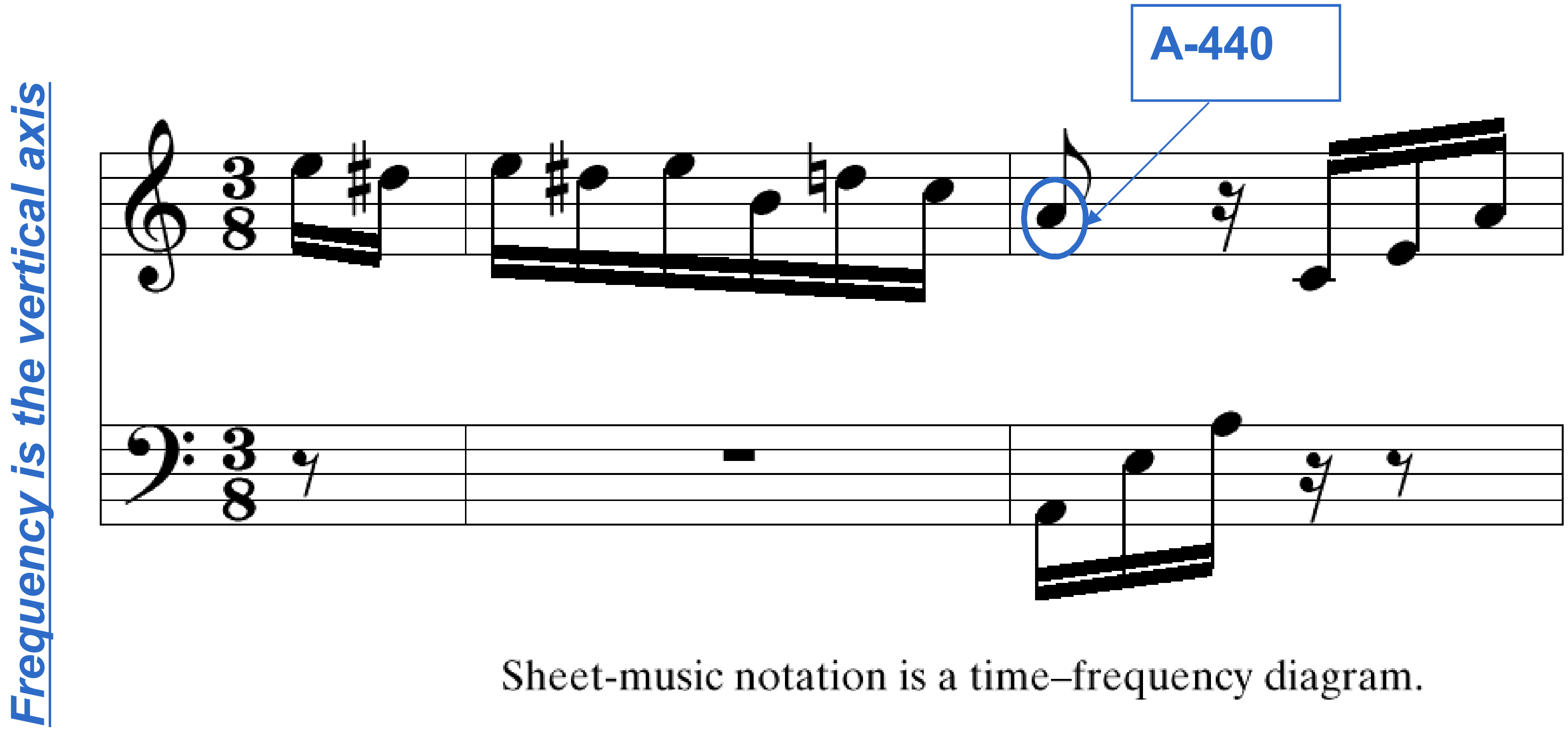






Spectrum Representation (1/2)

- Motivation



Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

Spectrum Representation (2/2)

- Euler's Formula

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

- Euler's Formula Reversed

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Spectrum Interpretation

- Cosine or sine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative frequency
Amplitude of each is half as big

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

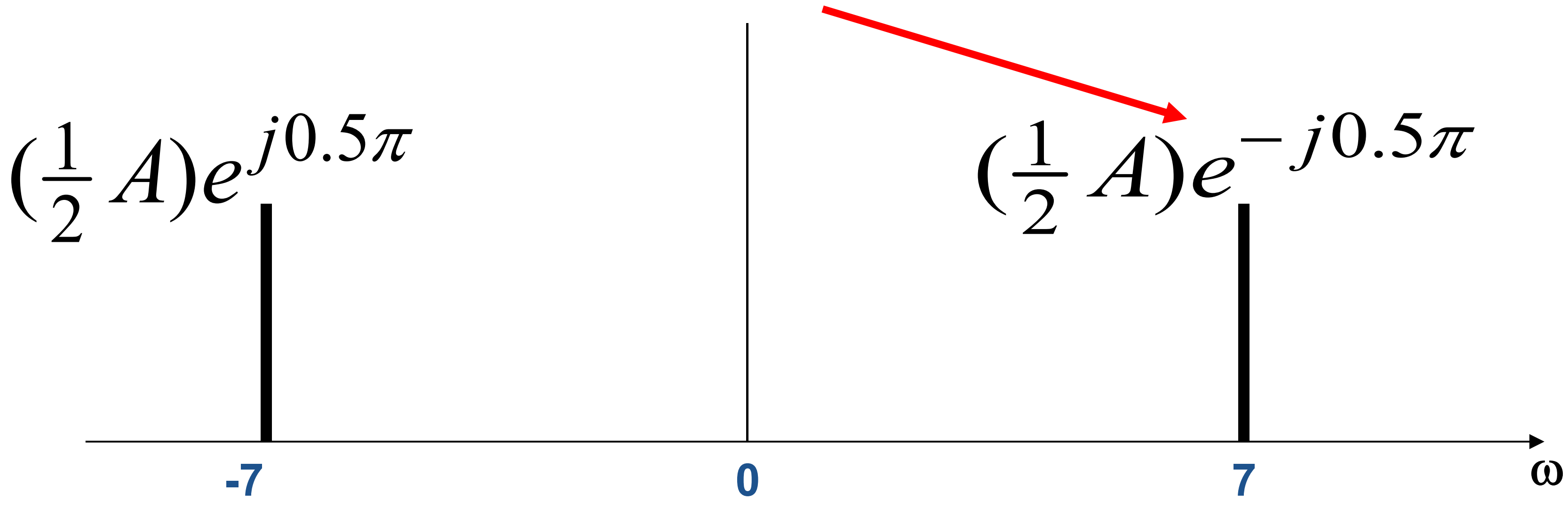
$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

$$\frac{-1}{j} = j = e^{j0.5\pi}$$

Graphical Spectrum (1/2)

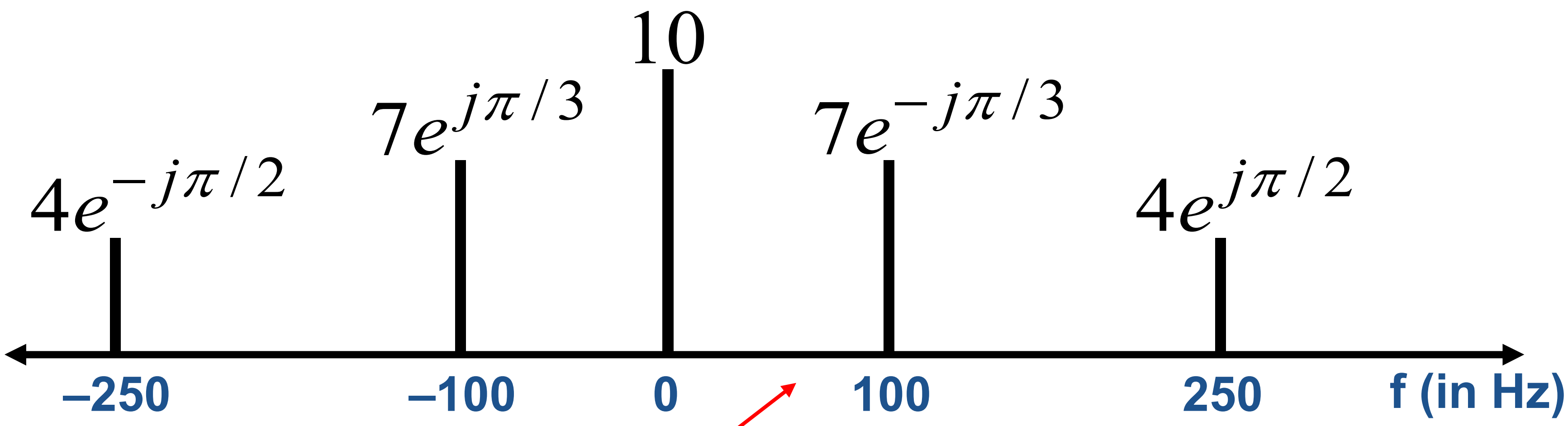
- Example of Sine

$$A\sin(7t) = \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$



Amplitude, Phase & Frequency are shown

Graphical Spectrum (2/2)



$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$+ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{-j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Summary: General Form

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

Harmonic Signal Spectrum (1/2)

Define Fundamental Frequency

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

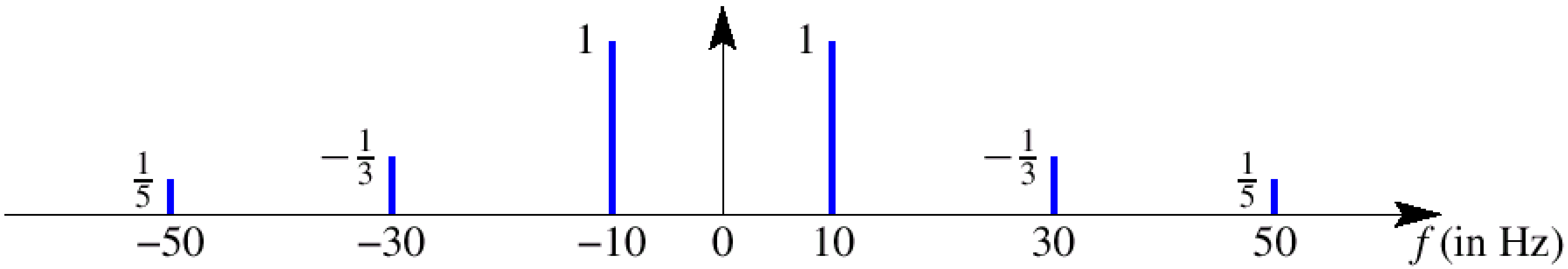
$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

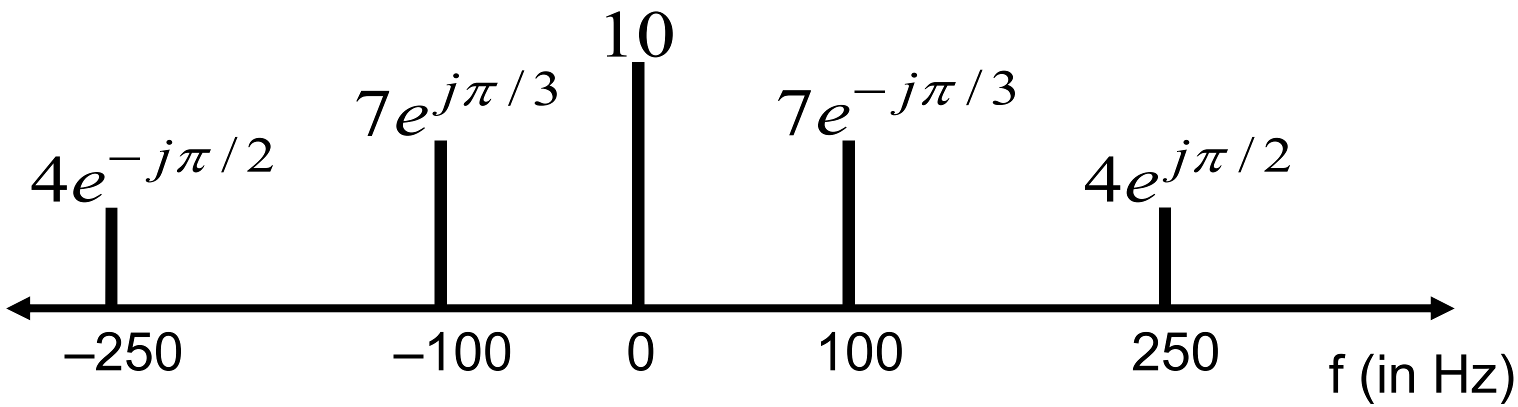
$$f_0 = \text{fundamental Frequency} \qquad f_0 = \frac{1}{T_0}$$

$$T_0 = \text{fundamental Period}$$

Harmonic Signal Spectrum (1/2)

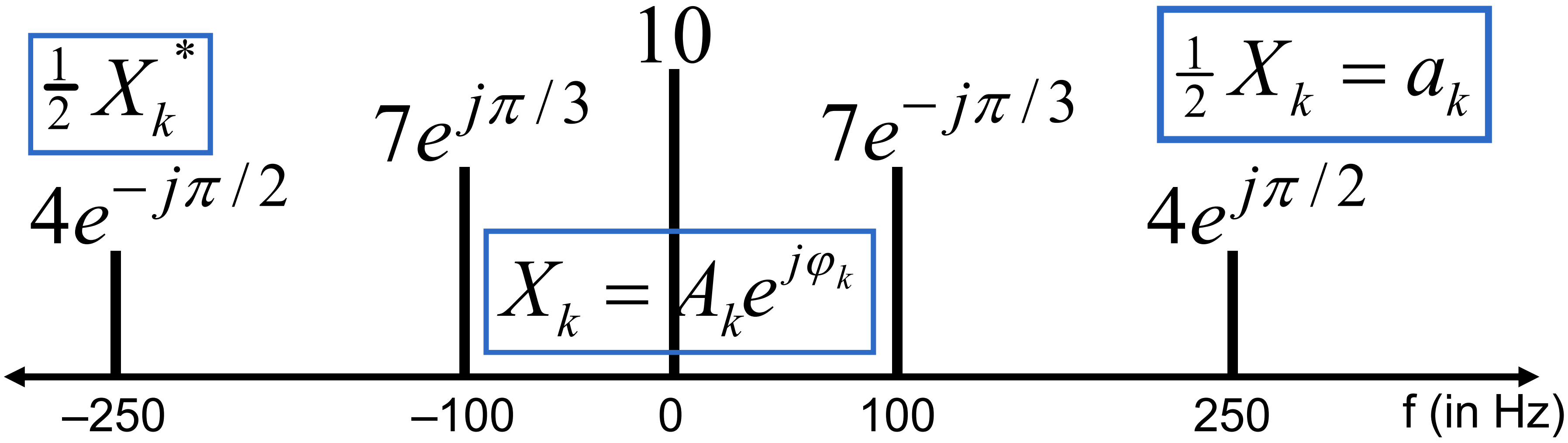
What is the fundamental frequency?





Spectrum Diagram

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

Fourier Series

- Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

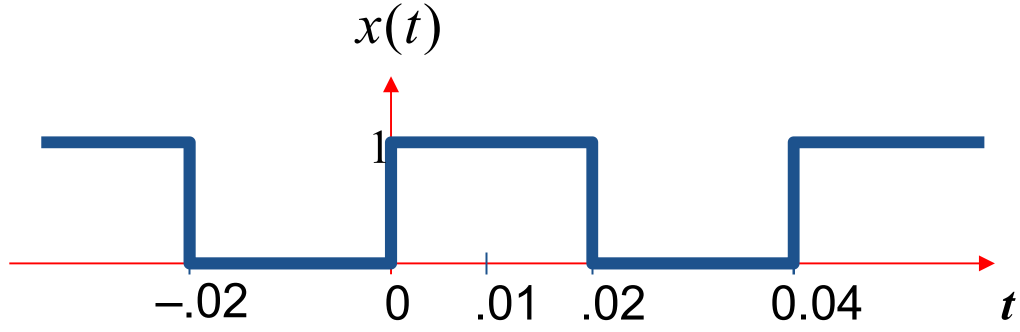
- Analysis

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

Example: Square Wave (1/3)

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \qquad (k \ne 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

Example: Square Wave (2/3)

DC Coefficient:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1\,dt = \frac{1}{.04}(.02-0) = \tfrac{1}{2}$$

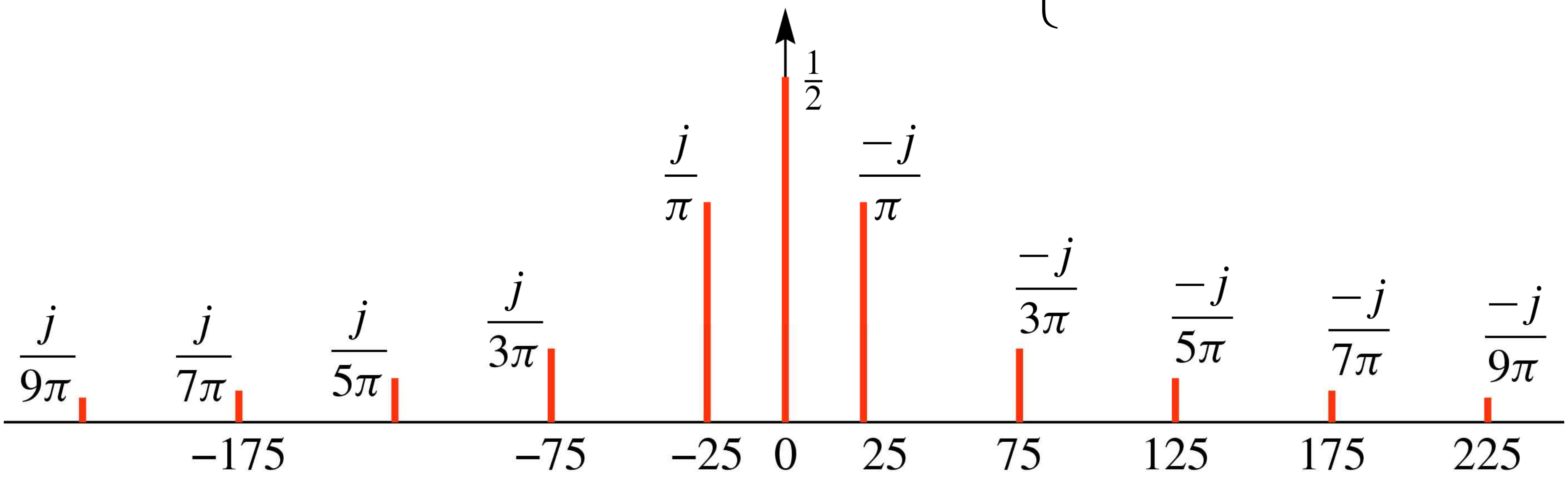
$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} -\dfrac{j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \tfrac{1}{2} & k = 0 \end{cases}$$

Example: Square Wave (3/3)

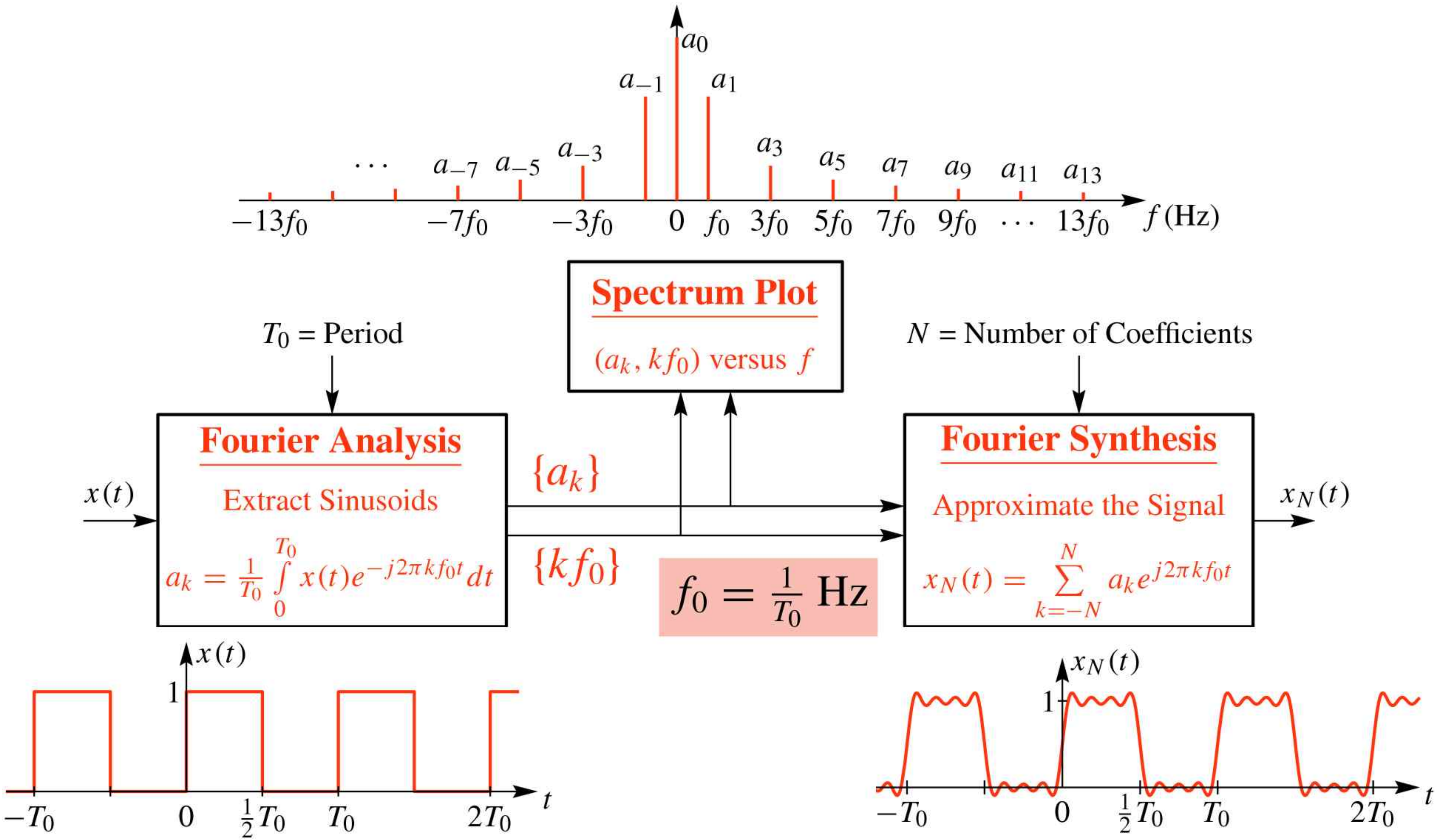
Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Summary



a_0
a_{-1} a_1
a_{-3} a_3
$\cdots$ a_{-7} a_{-5} a_5 a_7 a_9 a_{11} a_{13}
$-13f_0$ $-7f_0$ $-3f_0$ 0 f_0 $3f_0$ $5f_0$ $7f_0$ $9f_0$ $\cdots$ $13f_0$ f (Hz)
Spectrum Plot
(a_k, kf_0) versus f
T_0 = Period
N = Number of Coefficients
Fourier Analysis
Extract Sinusoids
$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k f_0 t}dt$
$x(t)$
$\{a_k\}$
$\{kf_0\}$
$f_0 = \frac{1}{T_0}$ Hz
Fourier Synthesis
Approximate the Signal
$x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$
$x_N(t)$
$x(t)$
1
$-T_0$ 0 $\frac{1}{2}T_0$ T_0 $2T_0$ t
$x_N(t)$
1
$-T_0$ 0 $\frac{1}{2}T_0$ T_0 $2T_0$ t