신호 및 시스템

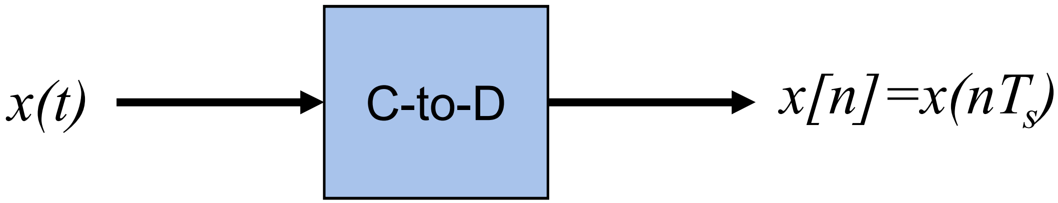
II. 샘플링과 에일리어싱

학습에 앞서

- 학습목표
  - 연속시간 영역과 디지털영역 사이의 신호 변환에 대해서 샘플링 이론을 이해한다.
  - 에일리어싱 개념을 이해한다.

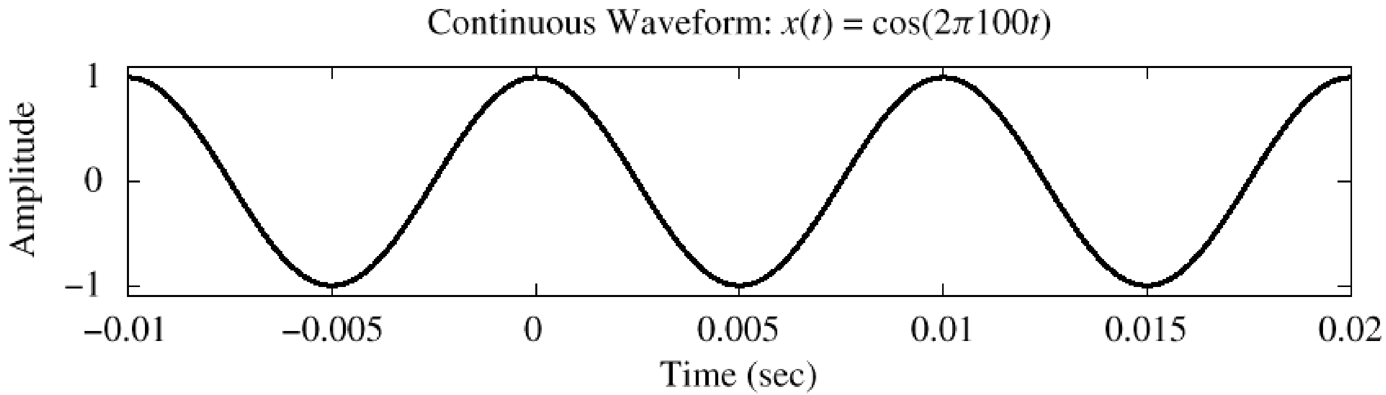
Sampling

- Sampling Process
  - Convert *x(t)* to numbers *x[n]*
  - "*n*" is an integer; *x[n]* is a sequence of values
  - Think of "n" as the storage address in memory
- Sampling Rate (f_s)
  - $f_s = 1/T_s$
    - Number of samples per second
    - Units: Hertz
    - Ex) T_s = 125 microsec → f_s = 8000 Hz
- Uniform sampling at $t = nT_s$
    - Ideal: $x[n] = x(nT_s)$

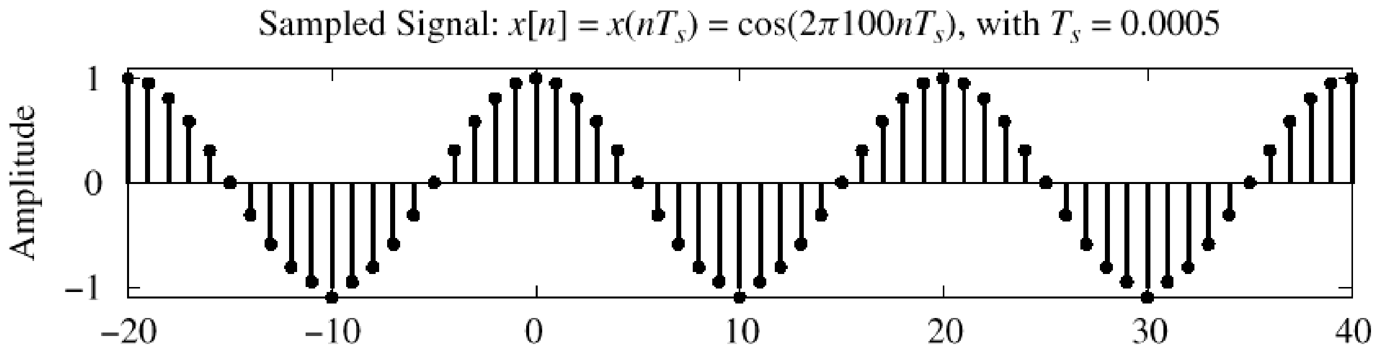


Sampling Rate

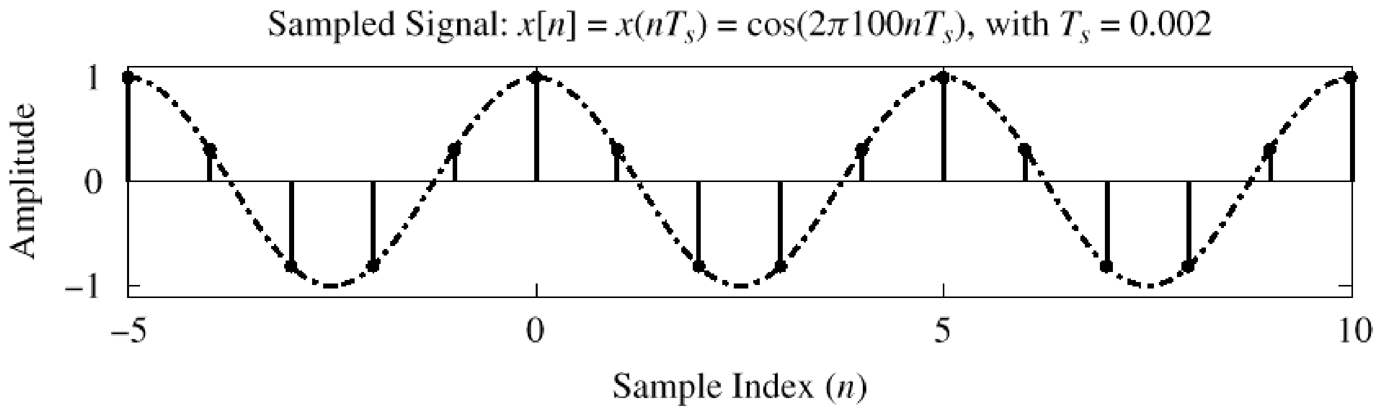
$f = 100\text{Hz}$



$f_s = 2\,\text{kHz}$



$f_s = 500\text{Hz}$



Sampling Theorem

- How often?
  - Depends on Frequency of Sinusoid
  - Answered by Shannon/Nyquist Theorem
  - Also depends on "**<u>Reconstruction</u>**"

Shannon Sampling Theorem

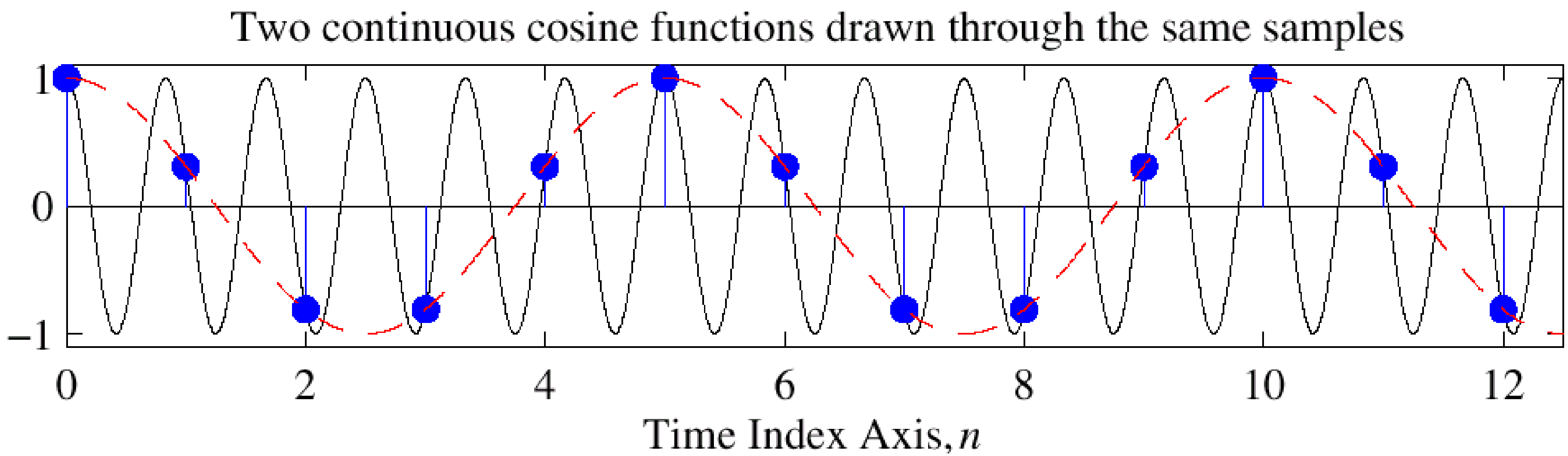
A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

Nyquist rate

Reconstruction

- Reconstruction? Which One?

Given the samples, draw a sinusoid through the values



When n is an integer

$$x[n] = \cos(0.4\pi n)$$

$$\cos(0.4\pi n) = \cos(2.4\pi n)$$

Discrete-time Sinusoid

- Change x(t) into x[n]

$$x(t) = A\cos(\omega t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega n T_s + \varphi) = A\cos((\omega T_s)n + \varphi)$$

- Digital frequency, $\hat{\omega}$

$$x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s} = \frac{2\pi f}{f_s}$$

  - Varies from **0** to **2π**, as f varies from 0 to the sampling frequency
  - Units are radians, **not** rad/sec
  - Digital frequency is normalized radian frequency

Aliasing (1/2)

- Other Frequencies give the same

$$x_1(t) = \cos(400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_1[n] = \cos(400\pi \tfrac{n}{1000}) = \underline{\cos(0.4\pi n)}$$

$$x_2(t) = \cos(2400\pi t) \quad \text{sampled at } f_s = 1000\,\text{Hz}$$

$$x_2[n] = \cos(2400\pi \tfrac{n}{1000}) = \cos(2.4\pi n)$$

$$= \cos(0.4\pi n + 2\pi n) = \underline{\cos(0.4\pi n)}$$

$$\Bigg\} \quad x_1[n] = x_2[n]$$

Aliasing (2/2)

- Other Frequencies give the same

$$\text{If } \; x(t) = A\cos(2\pi(\underline{f + \ell f_s})t + \varphi)$$

$$\text{and we want: } \; x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\text{then: } \hat{\omega} = \omega T_s$$

$$= \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$= \frac{2\pi f}{f_s} + \underline{2\pi\ell}$$

Aliasing

Aliasing: Folding

- Negative Frequencies can give the same

$$x_1(t) = A\cos(2\pi f t + \varphi)$$

$$x_1[n] = \underline{A\cos(\hat{\omega} n + \varphi)}$$

$$x_2(t) = A\cos(2\pi(-f + \ell f_s)t - \varphi)$$

$$x_2[n] = x(nT_s) = A\cos(2\pi(-f + \ell f_s)nT_s - \varphi)$$

$$= A\cos((-\hat{\omega} + 2\pi\ell)n - \varphi)$$

$$= A\cos(\hat{\omega} n + \varphi)$$

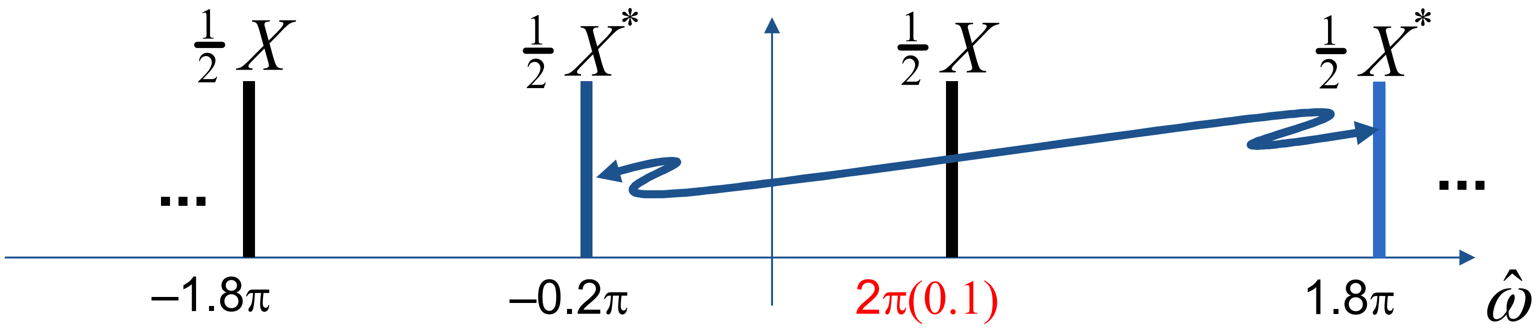
$$x_1[n] = x_2[n]$$

Spectrum for x[n]

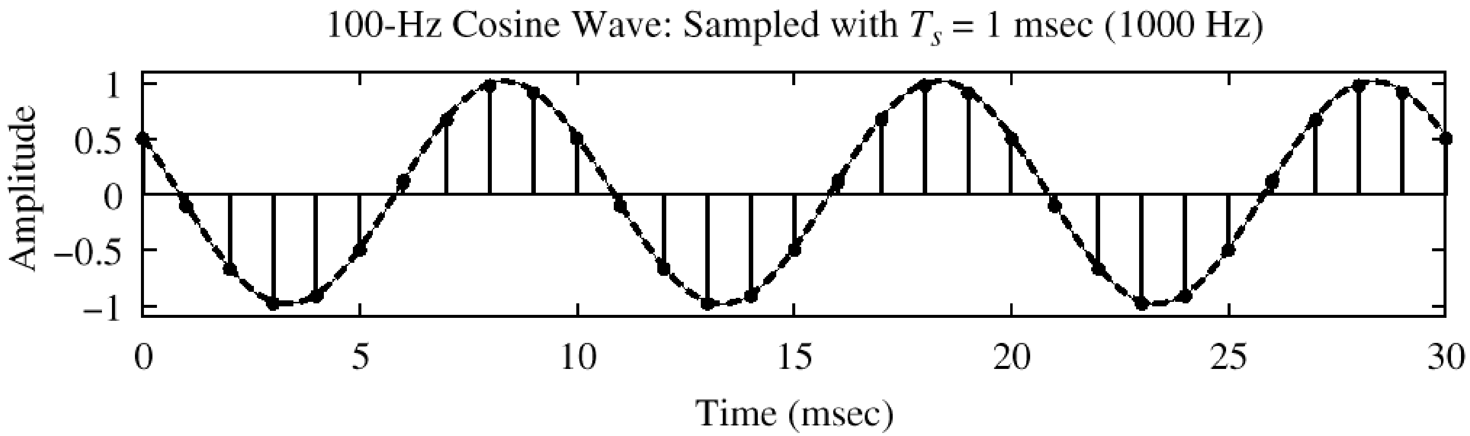
- Plot versus normalized frequency
- Include **ALL** spectrum lines
  - Aliases: add and subtract multiples of 2π
  - Folded aliases: aliases of negative frequencies

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 1\,\text{kHz}$$



$$x[n] = A\cos(2\pi(100)(n/1000) + \varphi)$$

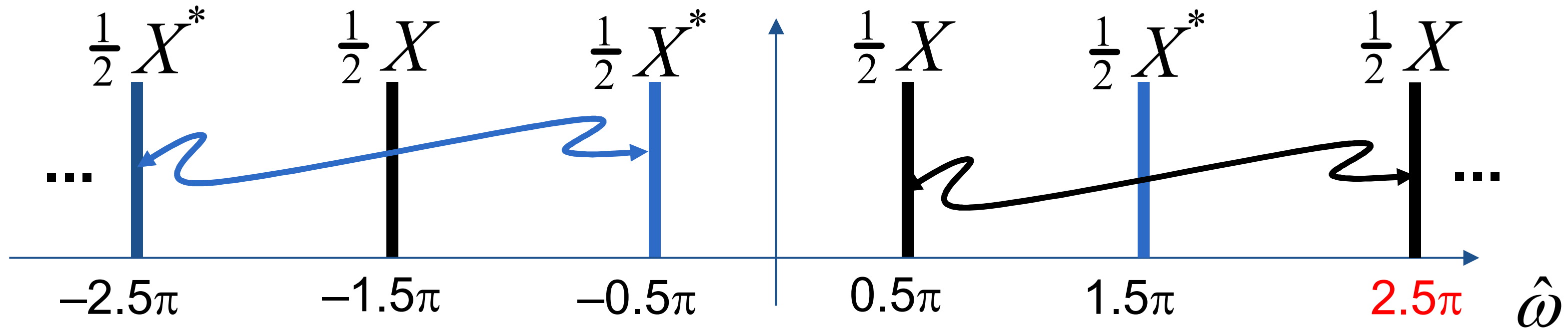


Spectrum (Aliasing Case)

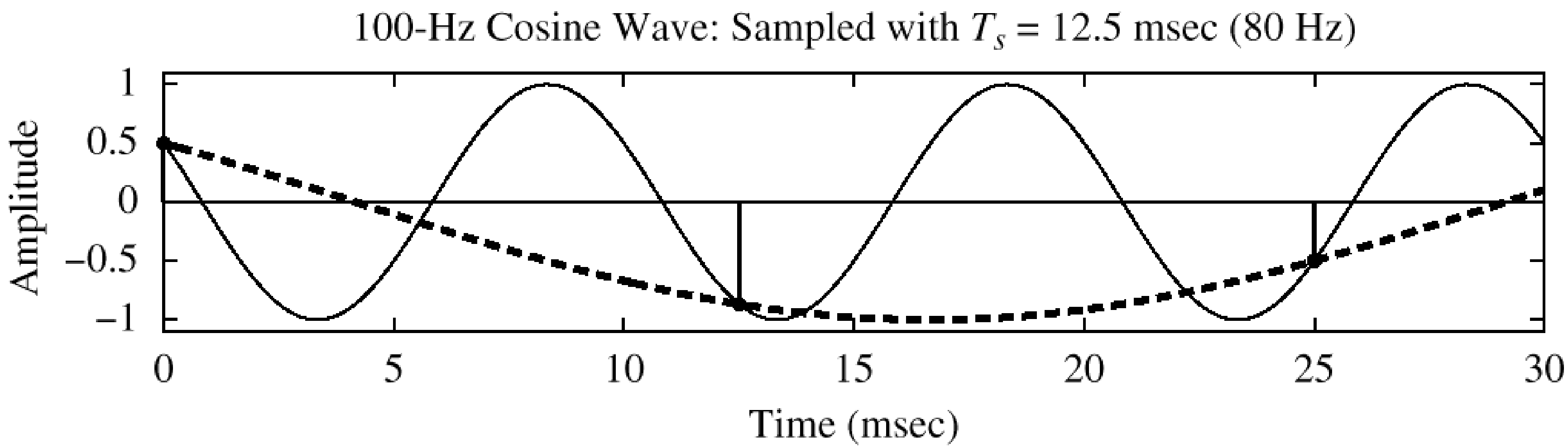
- Spectrum (Aliasing Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 80\,\text{kHz}$$



$$x[n] = A\cos(2\pi(100)(n/80) + \varphi)$$

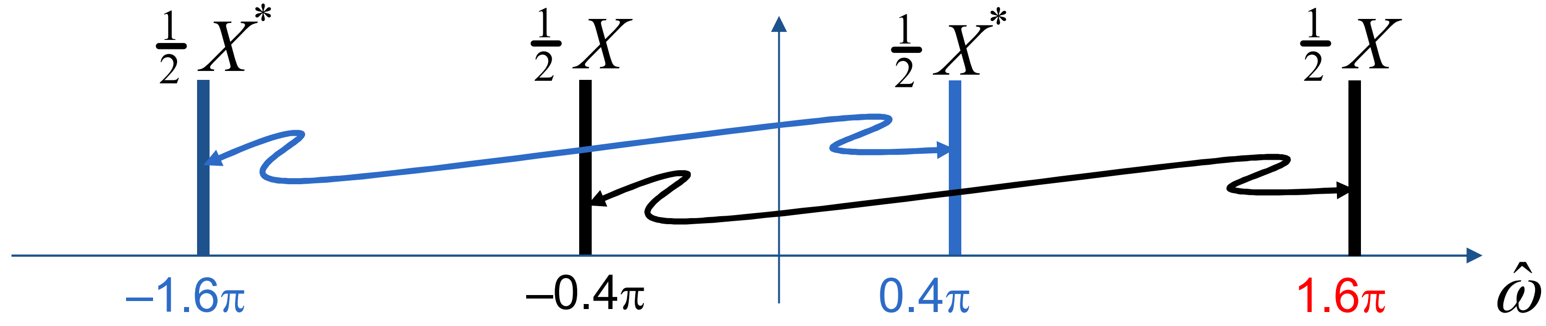


Spectrum (Folding Case)

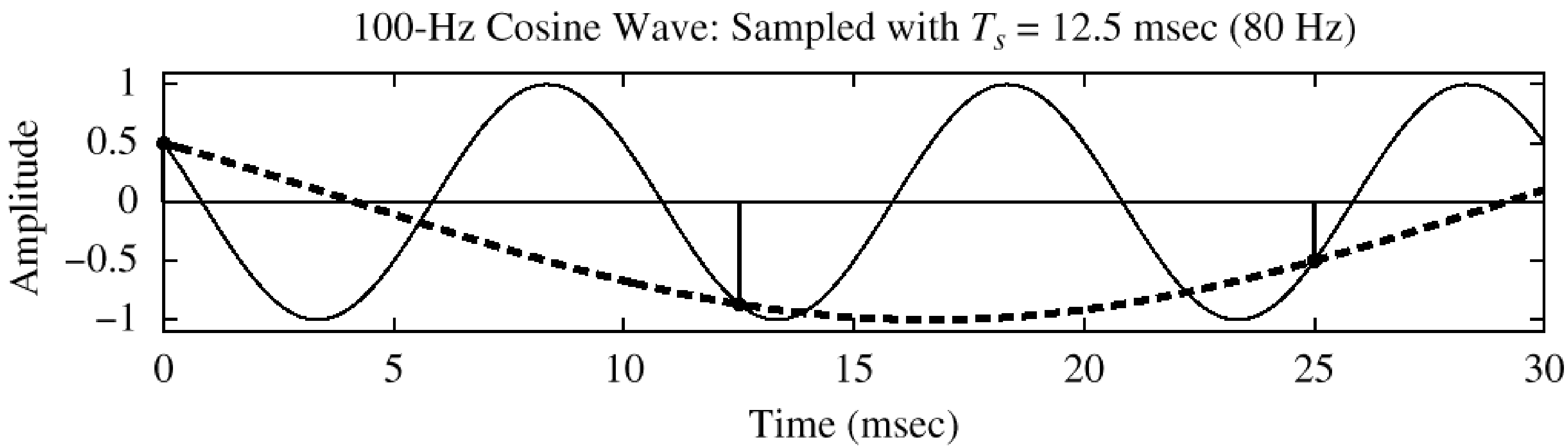
- Spectrum (Folding Case)

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

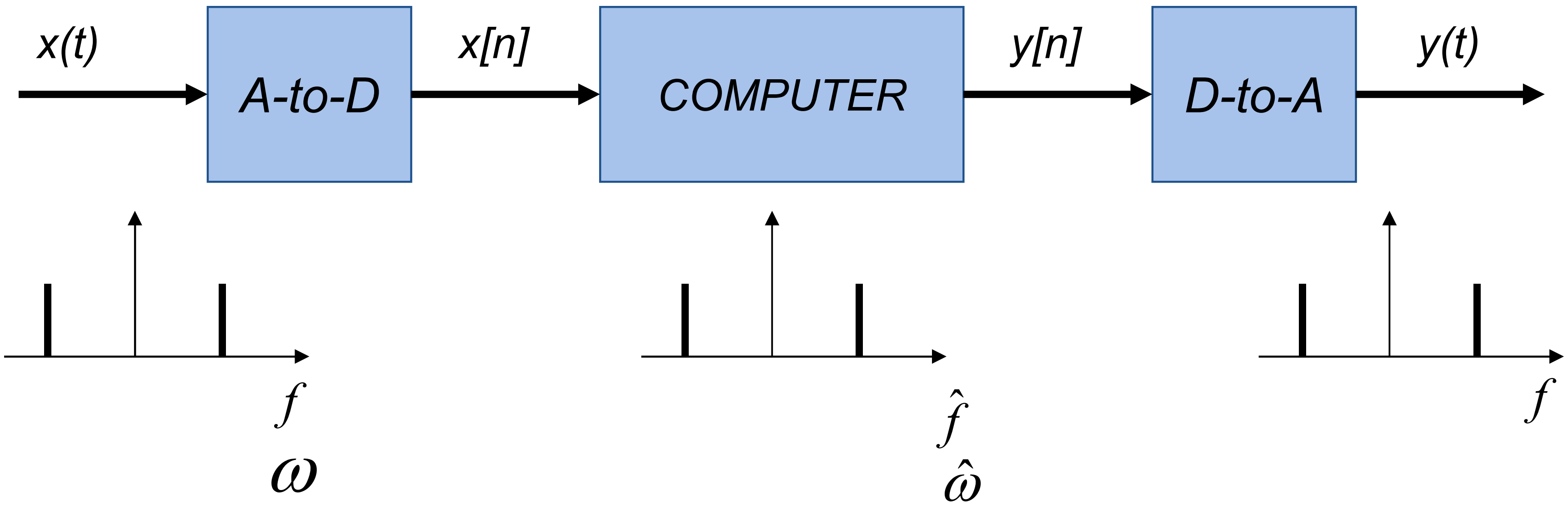
$$f_s = 125\text{Hz}$$



$$x[n] = A\cos(2\pi(100)(n/125) + \varphi)$$



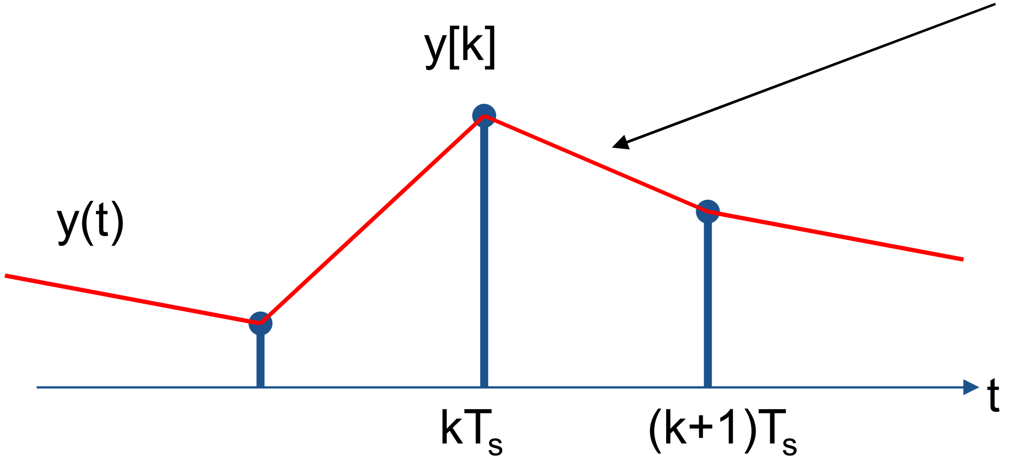
Frequency Domains



$$\hat{\omega} = 2\pi \frac{f}{f_s} + 2\pi\ell \qquad f = \frac{\hat{\omega}}{2\pi} f_s$$

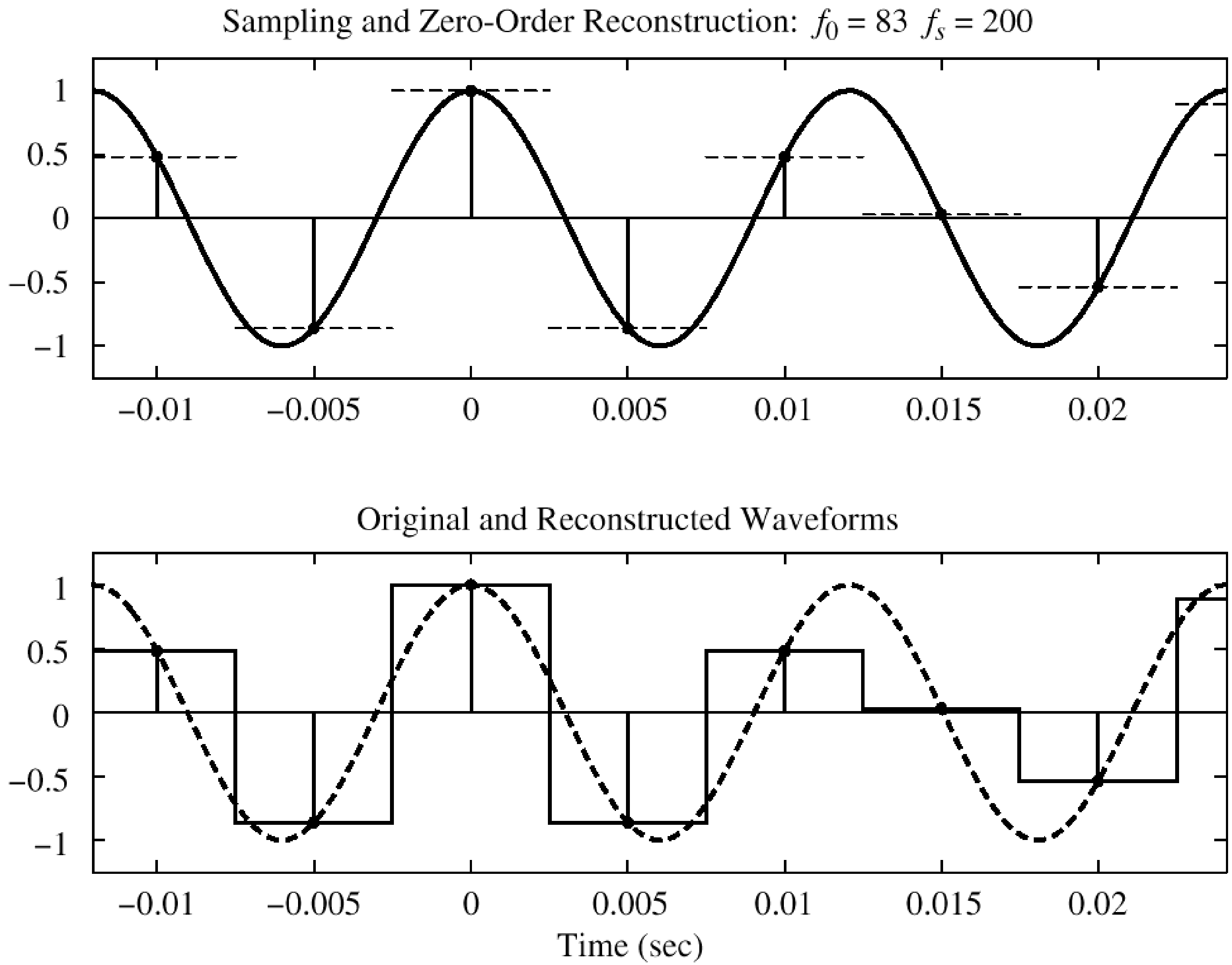
D-to-A Reconstruction

- Create continuous y(t) from y[n]
  - Select the lowest digital frequency, if y[n] = sinusoid
  - Ideal
    - If you have formula for y[n]
    - Replace n in y[n] with $f_s t$
    - Ex) $y[n] = A\cos(0.2\pi n+\phi)$ with $f_s = 8000$ Hz
      $y(t) = A\cos(2\pi(800)t+\phi)$
  - Practical
    - Connect the dots: interpolation



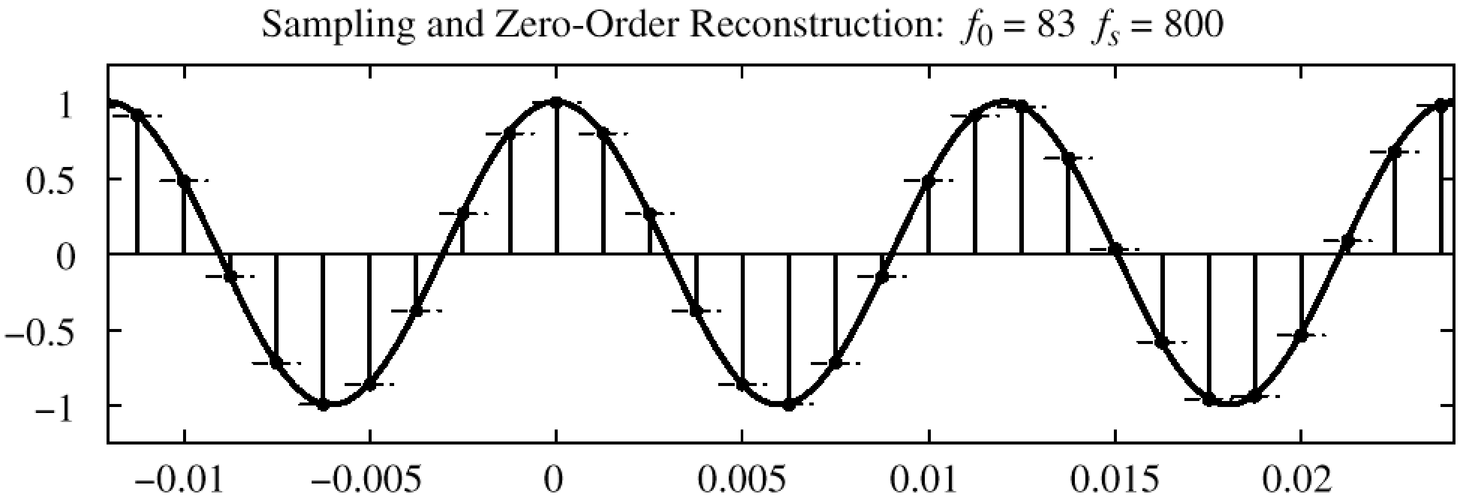
Reconstruction: Square Pulse Case

- By using Sample&Hold device

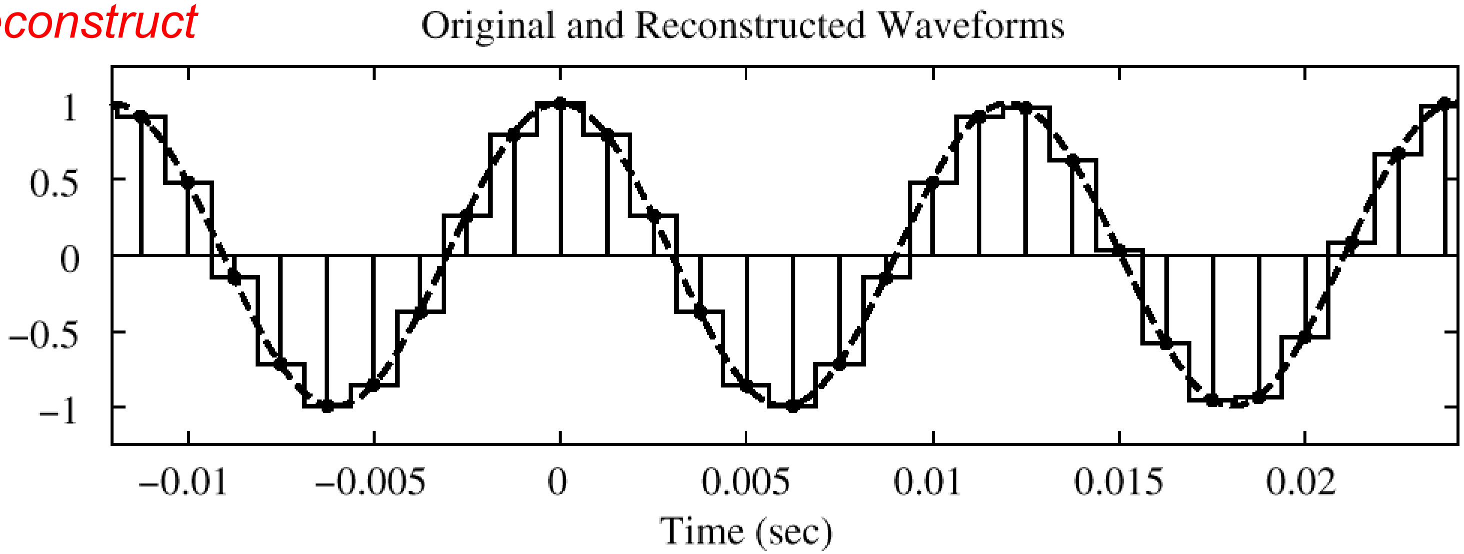


Reconstruction: Oversampling

- Oversampling



Easier to reconstruct



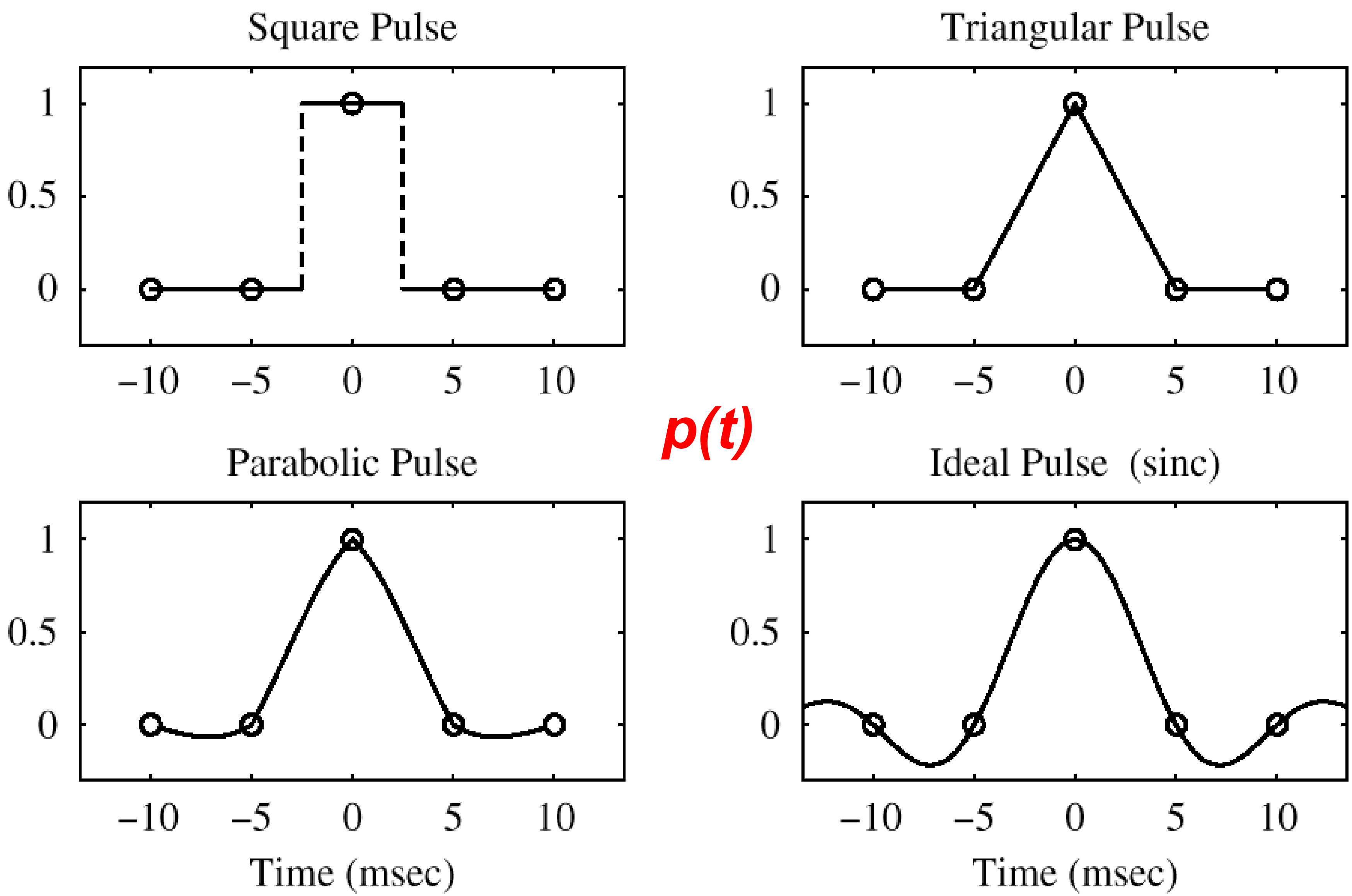
Mathematical Model for D-to-A (1/2)

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

Square Pulse $$p(t) = \begin{cases} 1 & -\frac{1}{2}T_s < t \leq \frac{1}{2}T_s \\ 0 & \text{otherwise} \end{cases}$$

Optimal Pulse $$p(t) = \begin{cases} \dfrac{\sin \frac{\pi t}{T_s}}{\frac{\pi t}{T_s}} & \text{for} -\infty < t < \infty \\ 0 & \text{for } t = \pm T_s, \pm 2T_s, \ldots \end{cases}$$

Mathematical Model for D-to-A (2/2)



Summary

- This lecture introduced the concept of sampling and the companion operation of reconstruction.

- Sampling Theorem
  - A continuous-time signal with frequencies no higher than f_{max} can be reconstructed exactly from its samples, if $f_s \geq 2f_{max}$.

- With sampling, the possibility of aliasing always exists.