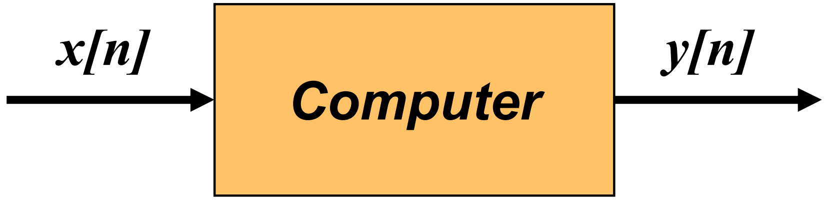
신호 및 시스템

III. FIR 필터

학습에 앞서

- 학습목표
  - FIR 필터의 개념 및 특성을 이해한다.
  - FIR 필터의 구현 방법을 학습한다.
  - FIR 필터의 주파수 응답을 구한다.

Discrete-time System



- Operate on x[n] to get y[n]
- A general class of systems
  - **ANALYZE** the system
    - Tools: time-domain & frequency-domain
  - **SYNTHESIZE** the system
- Example
  - Running average
    - RULE: “the output at time n is the average of three consecutive input values”

General FIR Filter

- Filter coefficients $\{b_k\}$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

  - Filter order is M
  - Filter length is L = M+1
  - Number of filter coefficients is L

- Example
  - 3-pt Average System $y[n] = \frac{1}{3}(x[n] + x[n+1] + x[n+2])$
  - $b_k = \{3, -1, 2, 1\}$ $y[n] = \sum_{k=0}^{3} b_k x[n-k]$

$$= 3x[n] - x[n-1] + 2x[n-2] + x[n-3]$$

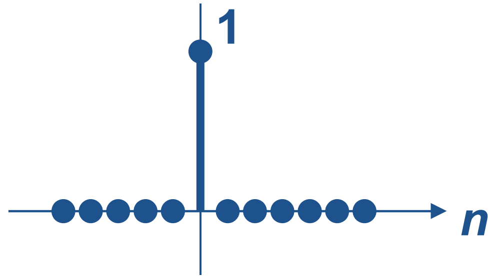
Example: filtered stock signal

- 50-pt Averager

Special Input Signals (1/2)

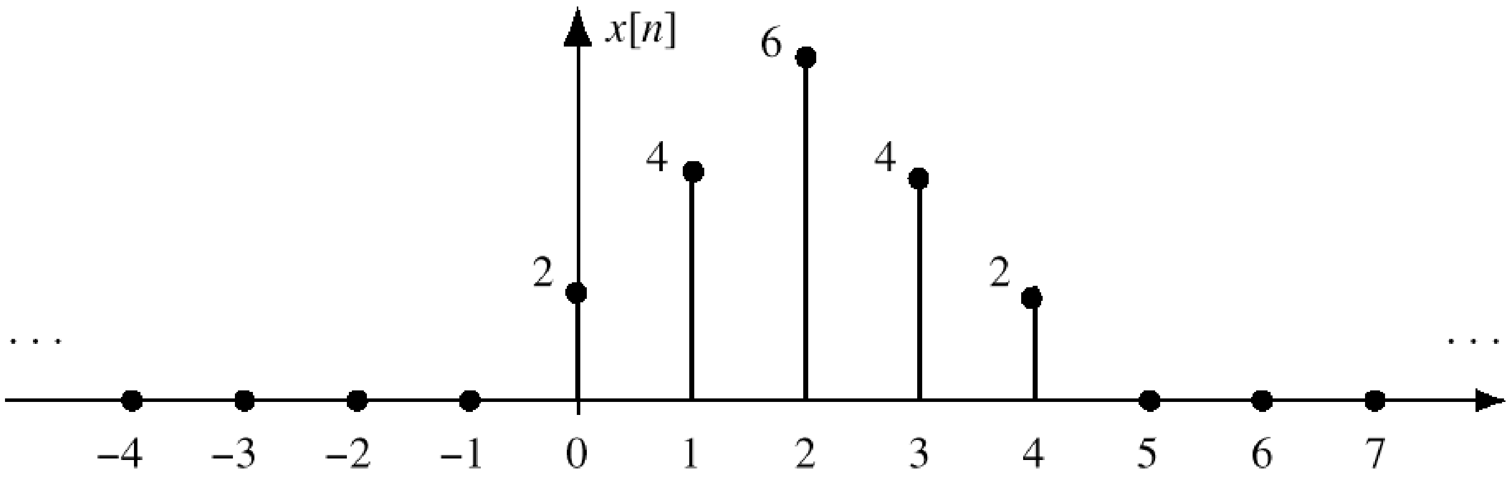
- Unit impulse
  - The mathematical notation is that of the Kronecker delta function.

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$$x[n] = \sum_k x[k]\delta[n-k] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

$$x[n] = 2\delta[n] + 4\delta[n-1] + 6\delta[n-2] + 4\delta[n-3] + 2\delta[n-4]$$



Special Input Signals (2/2)

- Unit Impulse response
  - When the input to the FIR filter is a unit impulse sequence, $x[n]=\delta[n]$, the output is, by definition, the unit impulse response, which is denoted by $h[n]$.

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]$$

$$h[n]=\sum_{k=0}^{M} b_k \delta[n-k]$$

⇒ Since h[n]=0 for n<0, the system is called a finite impulse response (FIR).

$$y[n]=\sum_{k=0}^{M} h[k]x[n-k]$$

⇒ Convolution

Convolution

- Convolution and FIR Filters
  - Notation:

$$y[n] = h[n] * x[n]$$

  - FIR case:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

Finite limits

Same as b_k

Finite limits

Example: Convolution

$$h[n]=\delta[n]-\delta[n-1]+2\delta[n-2]-\delta[n-3]+\delta[n-4]$$

$$x[n]=u[n]$$

$$y[n]=\sum_{k=0}^{M}h[k]x[n-k]$$

n	-1	0	1	2	3	4	5	6	7
$x[n]$	0	1	1	1	1	1	1	1	...
$h[n]$	0	1	-1	2	-1	1	0	0	0
$h[0]x[n]$	0	1	1	1	1	1	1	1	1
$h[1]x[n-1]$	0	0	-1	-1	-1	-1	-1	-1	-1
$h[2]x[n-2]$	0	0	0	2	2	2	2	2	2
$h[3]x[n-3]$	0	0	0	0	-1	-1	-1	-1	-1
$h[4]x[n-4]$	0	0	0	0	0	1	1	1	1
$y[n]$	0	1	0	2	1	2	2	2	...

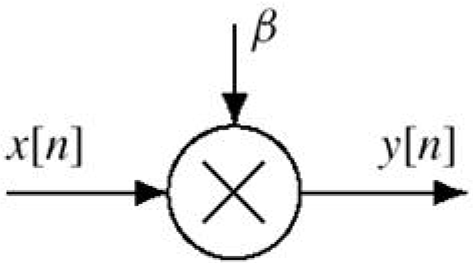
Implementation of FIR filters

- Recall the general definition of an FIR filter

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

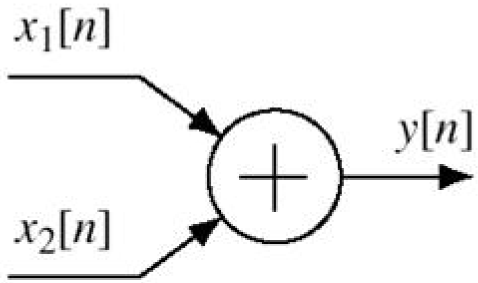
- The basic building-block systems we need are the multiplier, the adder, and the unit-delay operator.

$$y[n] = \beta x[n]$$



$$y[n] = x[n-1]$$

$$y[n] = x_1[n] + x_2[n]$$

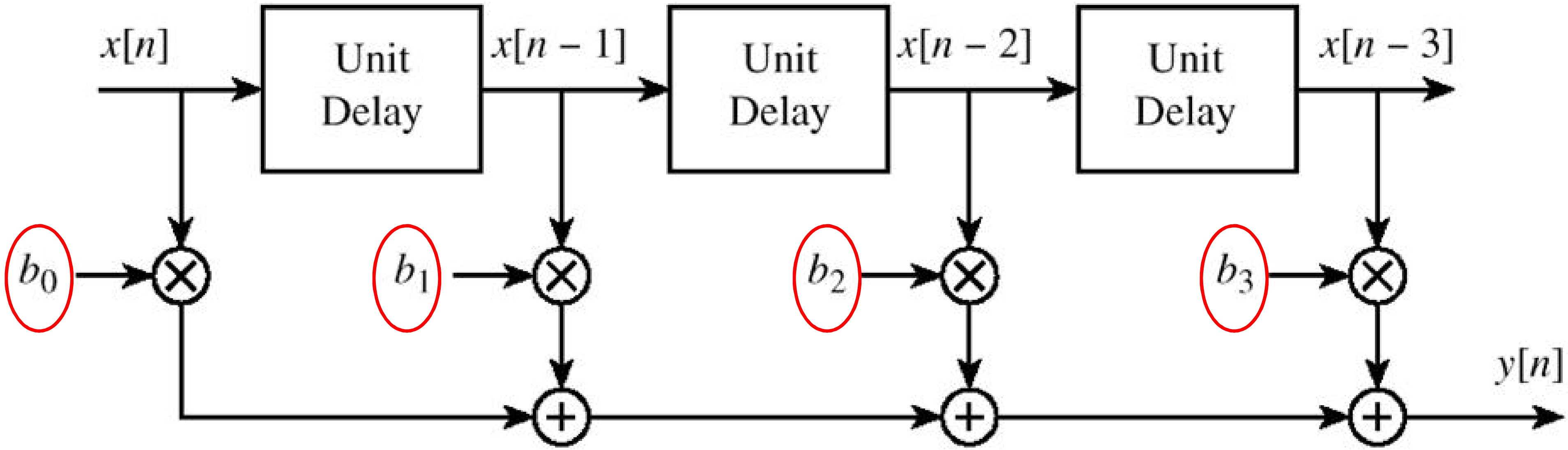


FIR Structure

- Direct Form

$$y[n]=\sum_{k=0}^{M} b_k x[n-k]$$

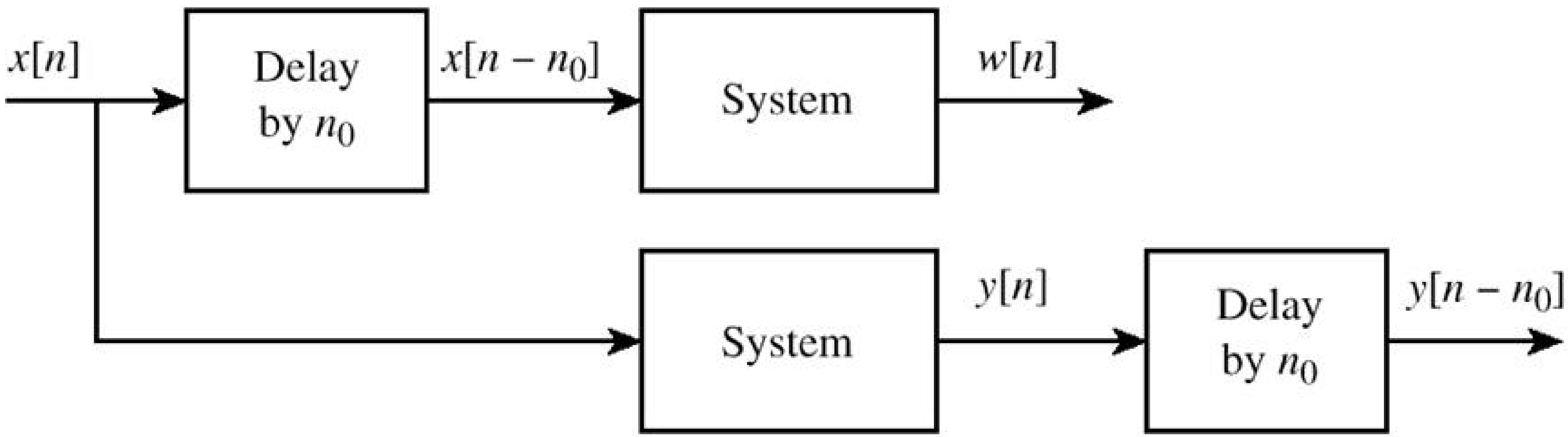
Signal flow graph



Block-diagram structure for the *M*th order FIR filter

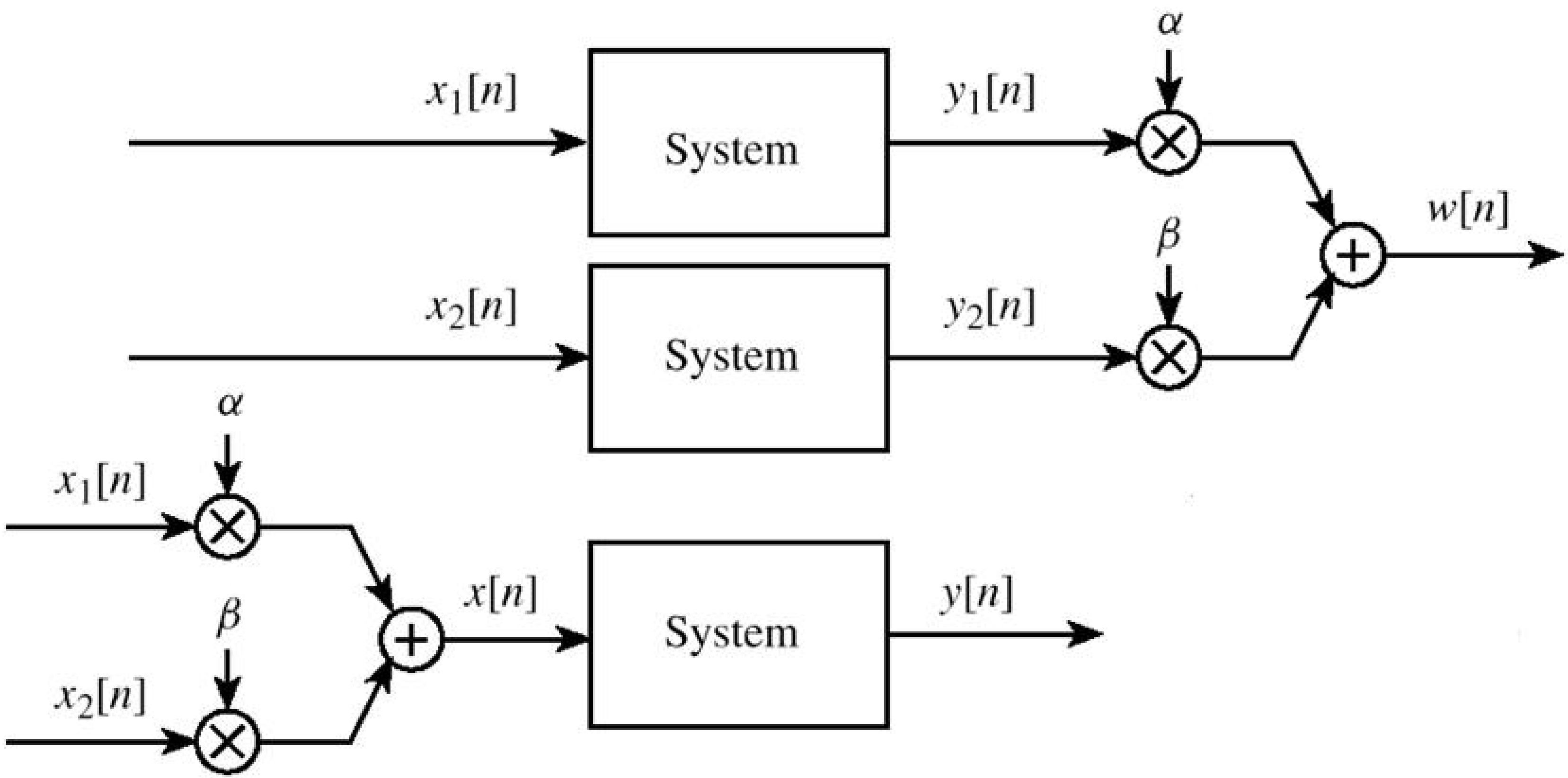
Linear Time-Invariant (LTI) Systems (1/2)

- Time Invariance
  – "Time-Shifting the input will cause the same time-shift in the output"



Linear Time-Invariant (LTI) Systems (2/2)

- Linearity = Two Properties
  - SCALING: "Doubling x[n] will double y[n]"
  - SUPERPOSITION: "Adding two inputs gives an output that is the sum of the individual outputs"

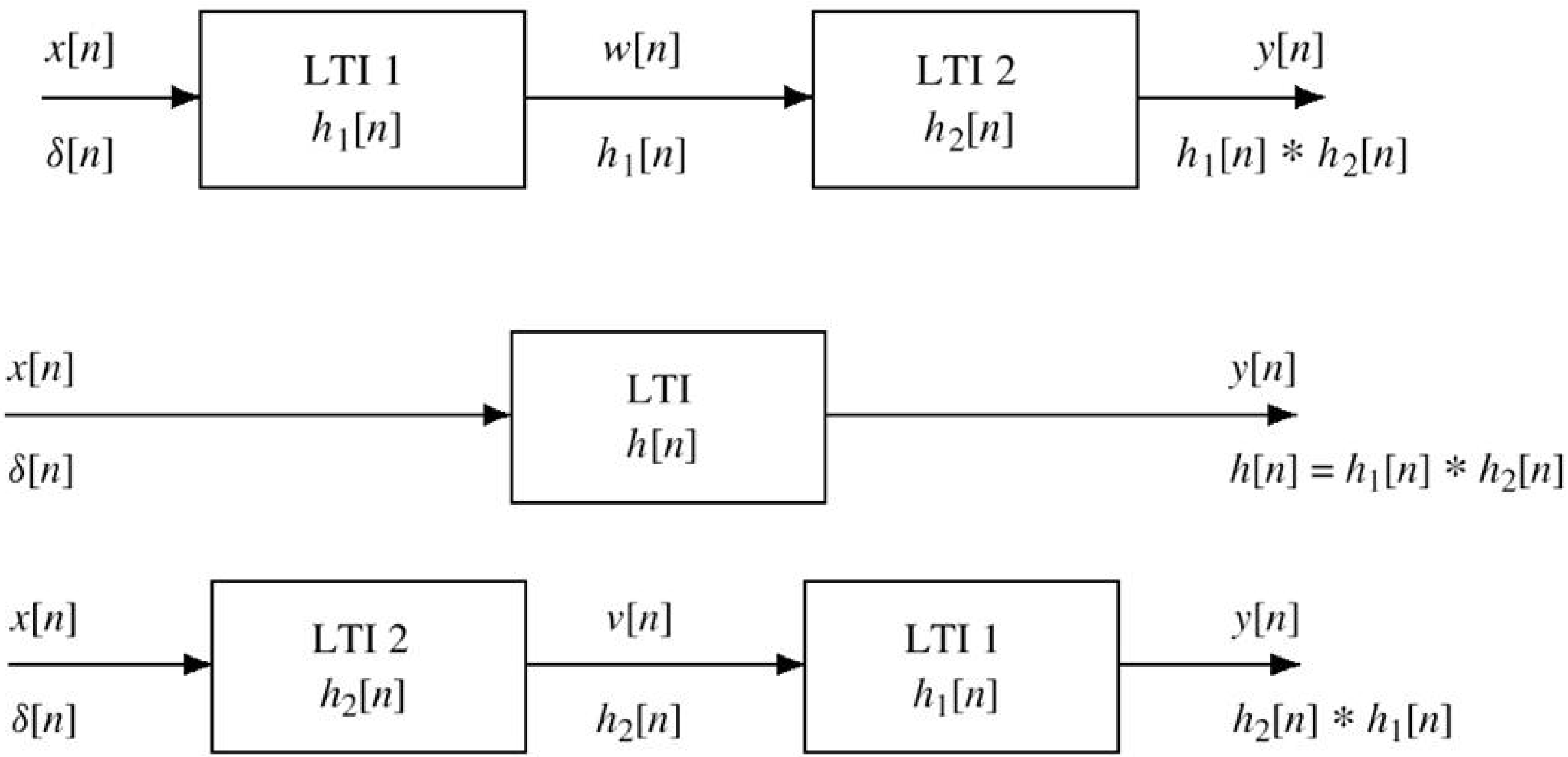


LTI systems

- Completely characterized by
  - Impulse response h[n]
  - Convolution y[n] = x[n]*h[n]
  - FIR Example: h[n] is same as b_k

- Properties of LTI systems
  - Commutative $x[n] * h[n] = h[n] * x[n]$
  - Associative $(x_1[n] * x_2[n]) * x_3[n] = x_1[n] * (x_2[n] * x_3[n])$

Cascaded LTI systems



Frequency Response of an FIR System

- Sinusoidal Response of FIR systems

$$x[n] = Ae^{j\varphi}e^{j\hat{\omega}n} \qquad -\infty < n < \infty$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k Ae^{j\varphi}e^{j\hat{\omega}(n-k)}$$

$$= \left(\sum_{k=0}^{M} b_k e^{j\hat{\omega}(-k)}\right) Ae^{j\varphi}e^{j\hat{\omega}n} = H(e^{j\hat{\omega}})x[n]$$

- Frequency-response

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$$

Example 1 (1/2)

- Show the frequency response of an FIR filter with coefficients

$$\{b_k\} = \{\,1,\ 2,\ 1\,\}$$

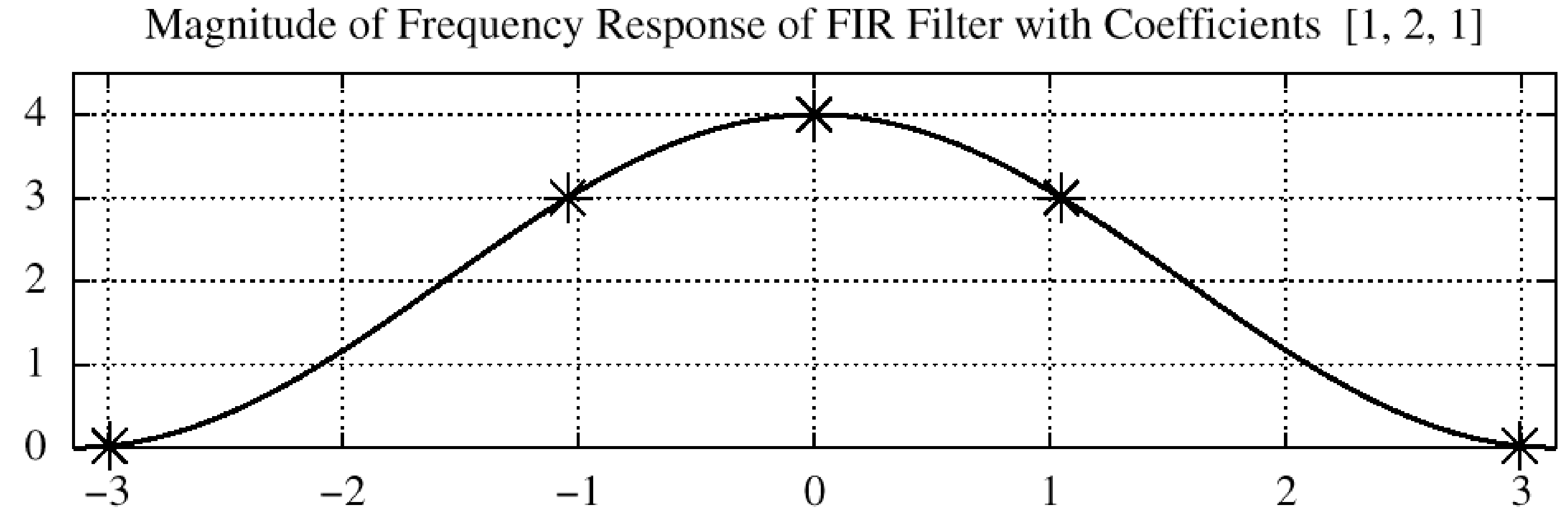
$$\begin{aligned} H(e^{j\hat{\omega}}) &= 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \\ &= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}) \\ &= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega}) \end{aligned}$$

Since $(2 + 2\cos\hat{\omega}) \geq 0$

Magnitude is $\left|H(e^{j\hat{\omega}})\right| = (2 + 2\cos\hat{\omega})$ and Phase is $\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$

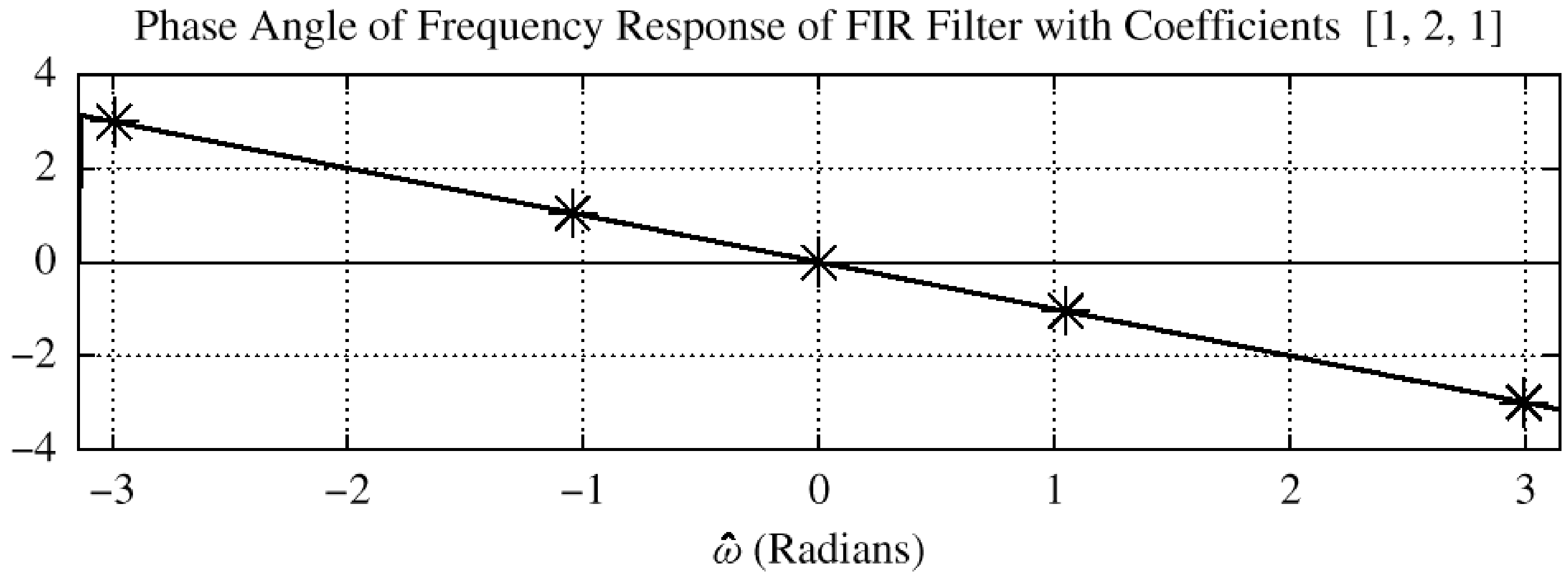
Example 1 (2/2)

$$\left|H(e^{j\hat{\omega}})\right| = (2 + 2\cos\hat{\omega})$$



$$\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$$

$$\mathcal{H}(\hat{\omega}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}.$$



Example 2

- Find y[n] for the following inputs when $H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$

1) $x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$

$$y[n] = \left(3e^{-j\pi/3}\right) \times 2e^{j\pi/4}e^{j(\pi/3)n} = 6e^{-j\pi/12}e^{j(\pi/3)n}$$

2) $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4}) = e^{j(\pi n/3 + \pi/4)} + e^{-j(\pi n/3 + \pi/4)}$

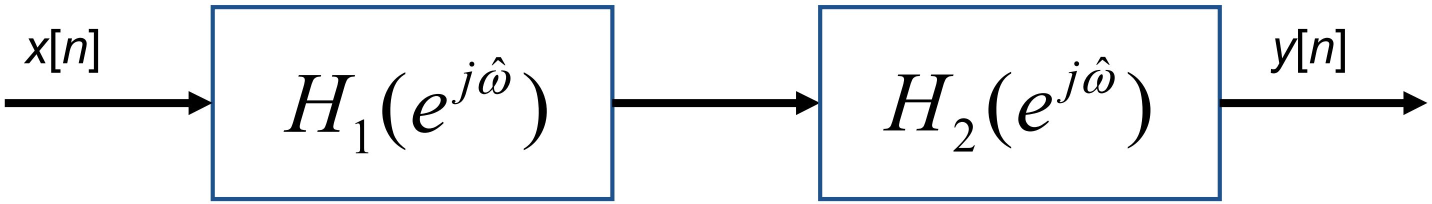
$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3 + \pi/4)} = 3e^{-j(\pi/3)}e^{j(\pi n/3 + \pi/4)}$$

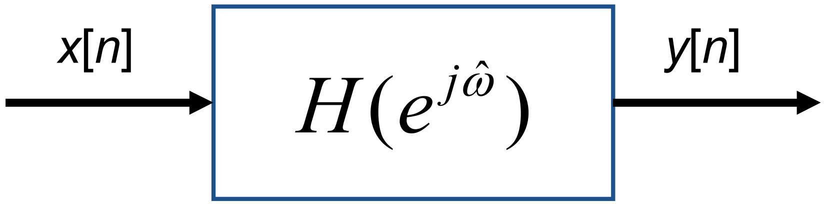
$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3 + \pi/4)} = 3e^{j(\pi/3)}e^{-j(\pi n/3 + \pi/4)}$$

$$y[n] = 3e^{j(\pi n/3 - \pi/12)} + 3e^{-j(\pi n/3 - \pi/12)} = 6\cos(\frac{\pi}{3}n - \frac{\pi}{12})$$

Cascaded LTI systems

- Multiply the frequency responses





  – Equivalent system

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$$

L-pt Averager

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}$$

$$= \frac{1}{L}\left(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}\right)$$

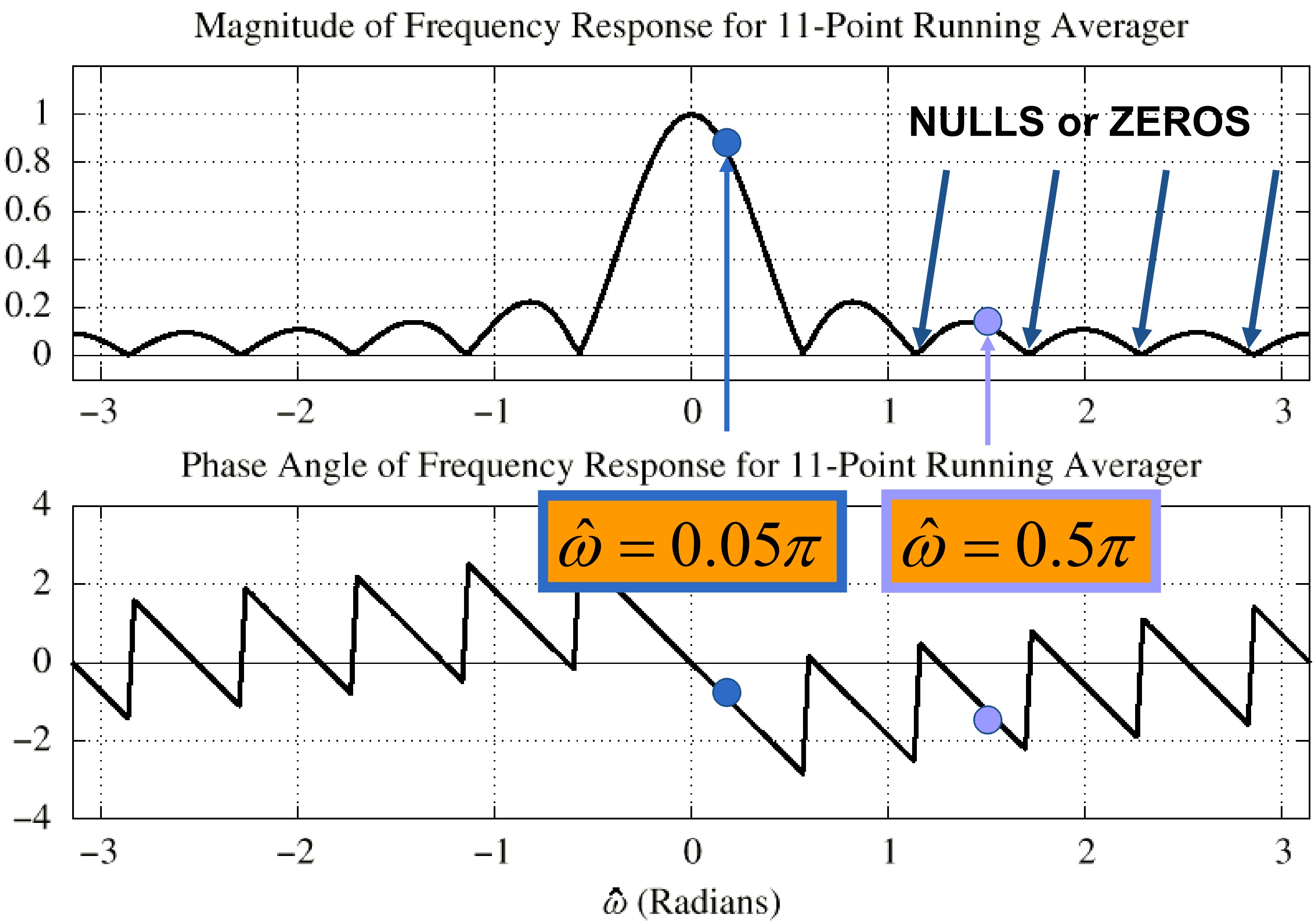
$$= \frac{1}{L}\left(\frac{e^{-j\hat{\omega}L/2}(e^{j\hat{\omega}L/2}-e^{-j\hat{\omega}L/2})}{e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2})}\right)$$

$$= \left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right)e^{-j\hat{\omega}(L-1)/2} = D_L(e^{j\hat{\omega}})e^{-j\hat{\omega}(L-1)/2}$$

- *Dirichlet* function

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

11-pt Averager

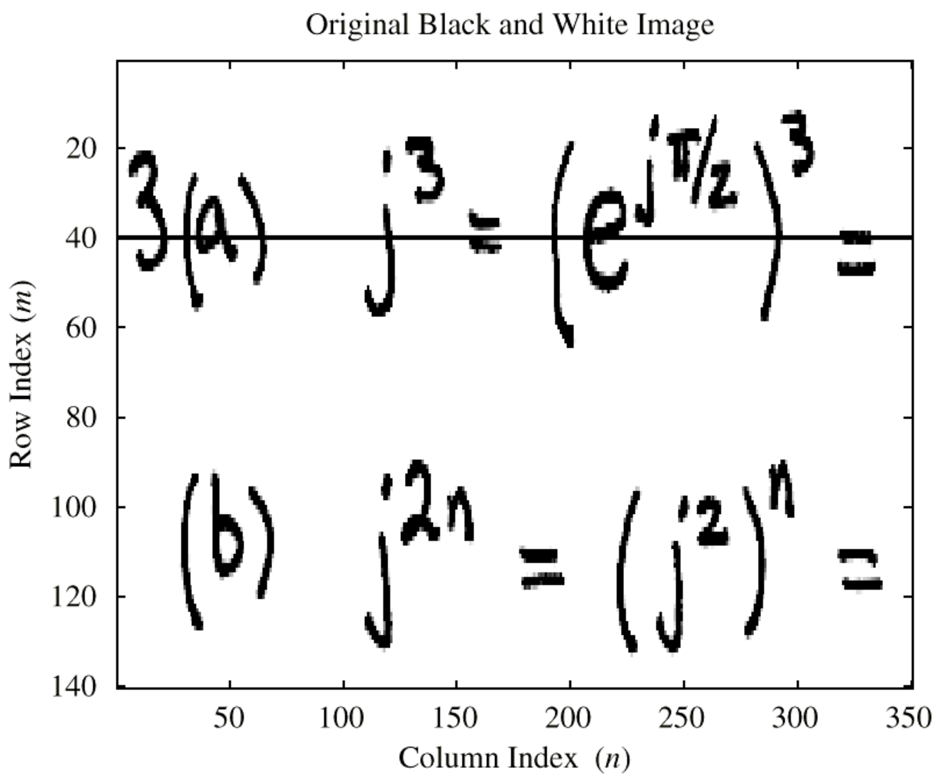


⇦ LPF

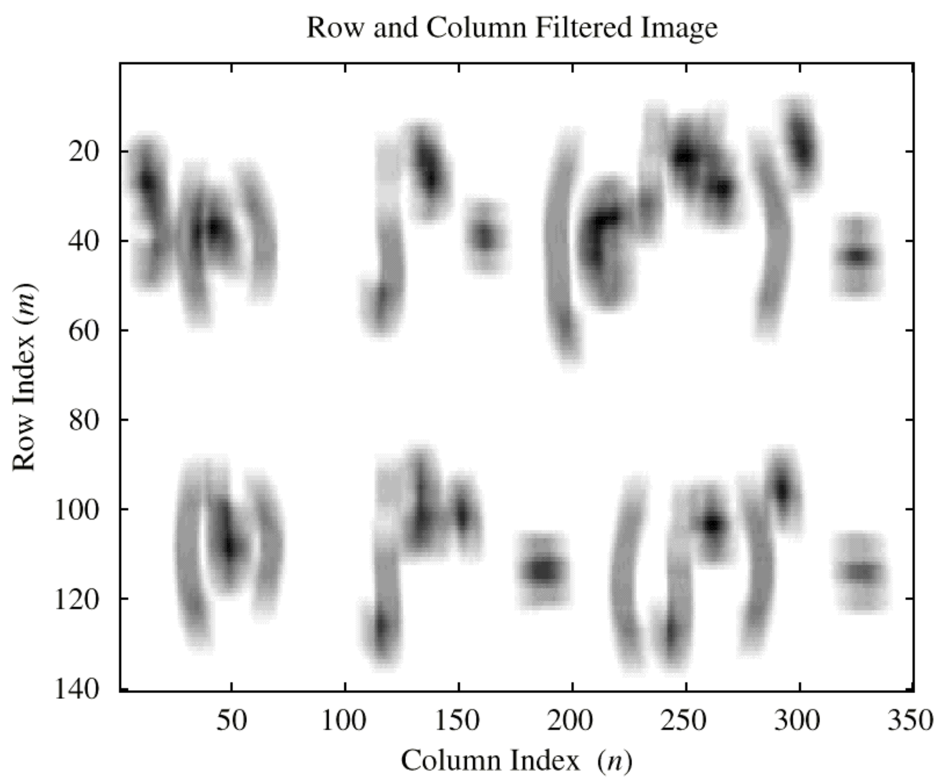
Example: L-pt Averager

- B&W Image

Original



Filtered by 11-pt averager



Summary

- This lecture introduced the concept of FIR filtering.
  - The weighted running average of a finite number of input sequence values defines a discrete-time system.

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

- The impulse response of an FIR system completely defines the system.

- The frequency-response function is a complete characterization of the behavior of the system for any input that can be represented as a sum of sinusoids.

$$y[n] = H(e^{j\hat{\omega}})x[n]$$