신호 및 시스템

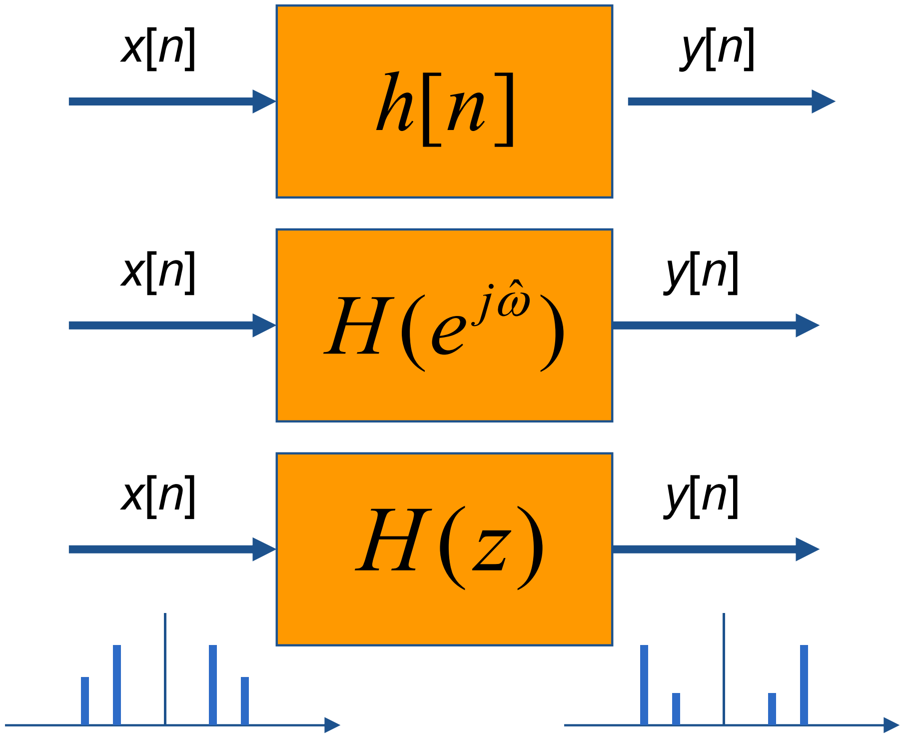
IV. z-Transforms

학습에 앞서

- 학습목표
  - z-변환의 필요성 및 방법을 이해한다.
  - z-변환의 활용을 학습한다.

Definition of the z-Transform

- Definition $$H(z) = \sum_n h[n] z^{-n}$$
- Move to a new domain where
  - Operations are easier and familiar
  - Use Polynomials



$$H(e^{j\hat{\omega}}) = \sum_n h[n] e^{-j\hat{\omega} n}$$

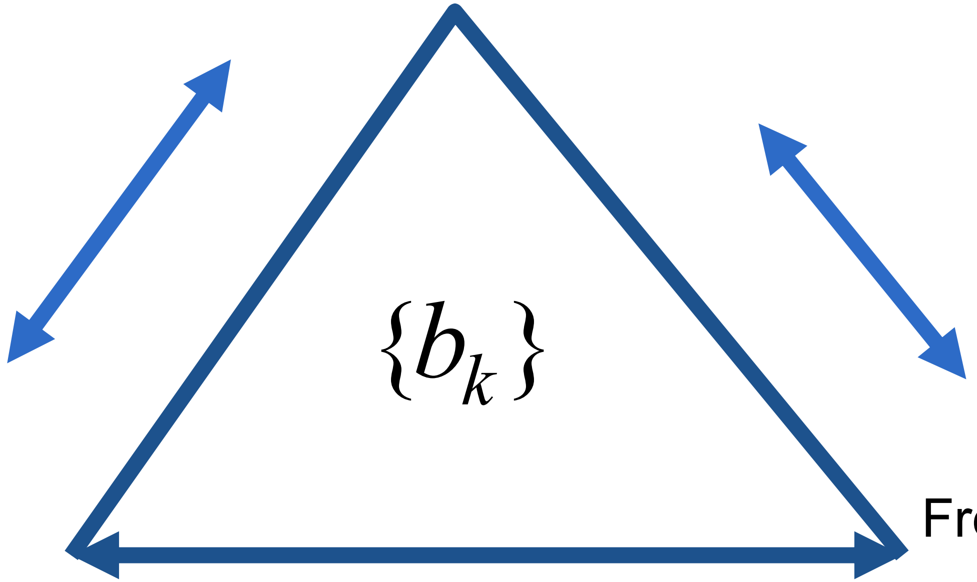
$$H(z) = \sum_n h[n] z^{-n}$$

Three Domains

Z-Transform-Domain

Polynomials: H(z)

$$H(z) = \sum_{n} h[n] z^{-n}$$



Time-Domain

Frequency-Domain

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

z-Transform of FIR Filter

- H(z) is called the system function.

System function

$$H(z)=\sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

$$y[n]=\sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k]$$

Difference Equation Convolution

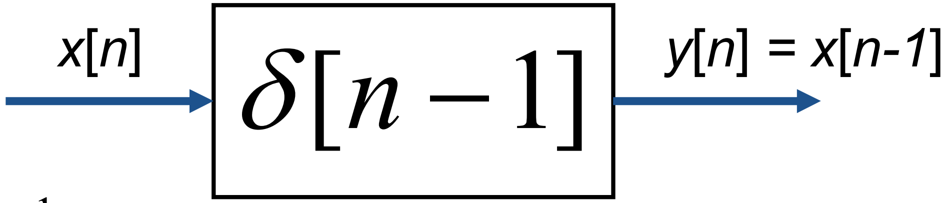
- Example

$$y[n]=6x[n]-5x[n-1]+x[n-2]$$

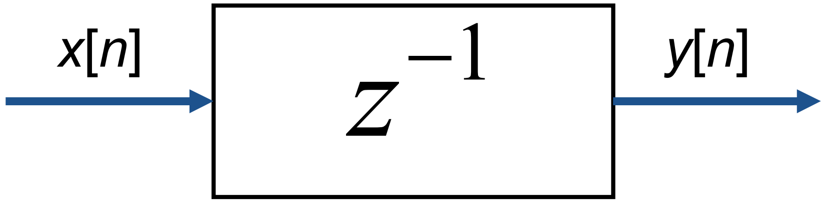
$$H(z)=\sum b_k z^{-1} = 6-5z^{-1}+z^{-2}$$

Unit-Delay Operator

- Unit-Delay Operator



$$H(z) = \sum \delta[n-1] z^{-n} = z^{-1}$$



A delay of one sample multiplies the z-transform by z^{-1}.

$$x[n-1] \quad \Longleftrightarrow \quad z^{-1} X(z)$$

Time delay of n_0 *samples multiplies the z-transform by* z^{-n_0}

$$x[n-n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

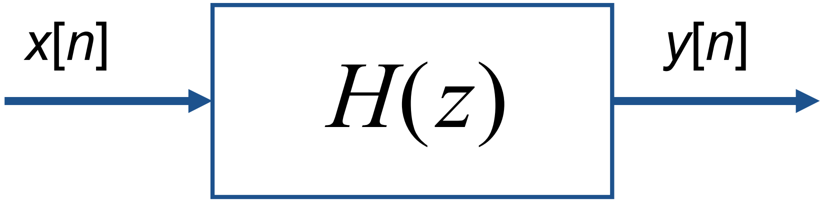
Convolution and the z-Transform

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$Y(z) = \sum_{k=0}^{M} h[k]\left(z^{-k}X(z)\right)$$

$$= \left(\sum_{k=0}^{M} h[k]z^{-k}\right)X(z) = H(z)X(z).$$

Convolution Example



$$x[n] = \delta[n-1] + 2\delta[n-2] \qquad h[n] = \delta[n] - \delta[n-1]$$

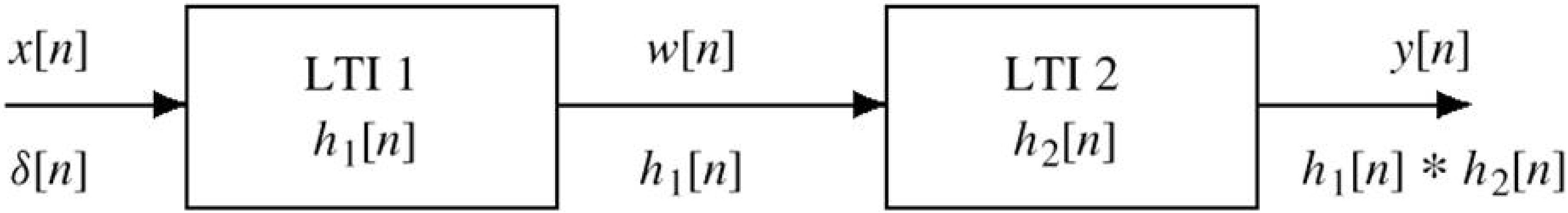
$$y[n] = x[n] * h[n]$$

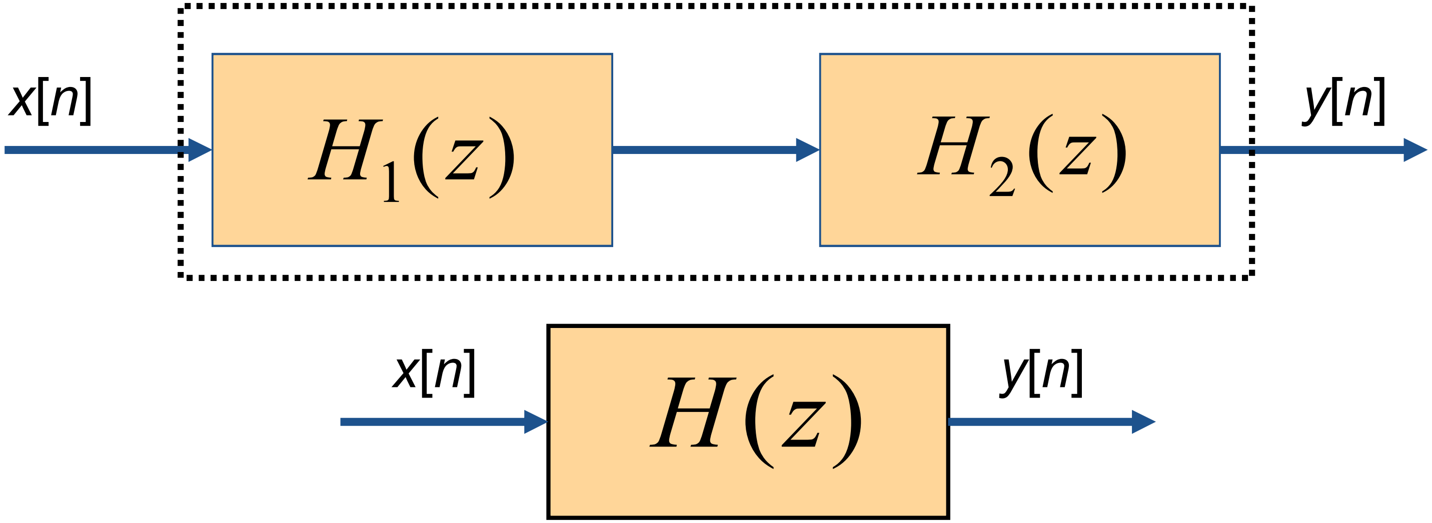
$$X(z) = z^{-1} + 2z^{-2} \qquad H(z) = 1 - z^{-1}$$

$$Y(z) = (z^{-1} + 2z^{-2})(1 - z^{-1}) = z^{-1} + z^{-2} - 2z^{-3}$$

$$y[n] = \delta[n-1] + \delta[n-2] - 2\delta[n-3]$$

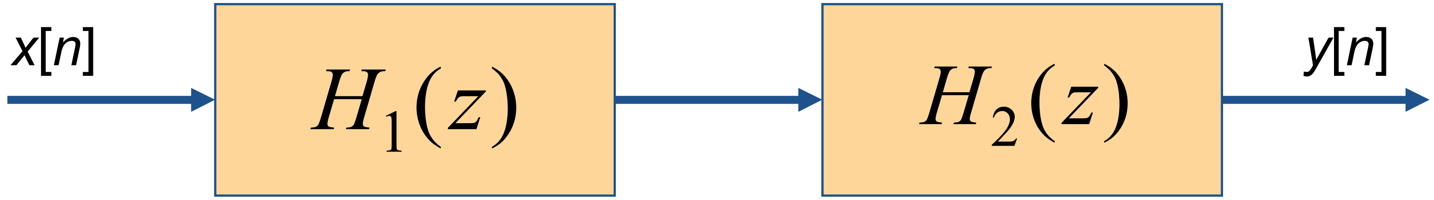
Cascading Systems





$$H(z) = H_1(z)H_2(z)$$

Deconvolution



$$Y(z) = H_1(z)H_2(z)X(z) = X(z)$$

$$H_1(z)H_2(z) = 1$$

$$H_2(z) = \frac{1}{H_1(z)}$$

Relationship Bet. the z-Domain and the $\hat{\omega}$ Domain (1/2)

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k (e^{j\hat{\omega}})^{-k}$$

$$z = e^{j\hat{\omega}}$$

z – Domain

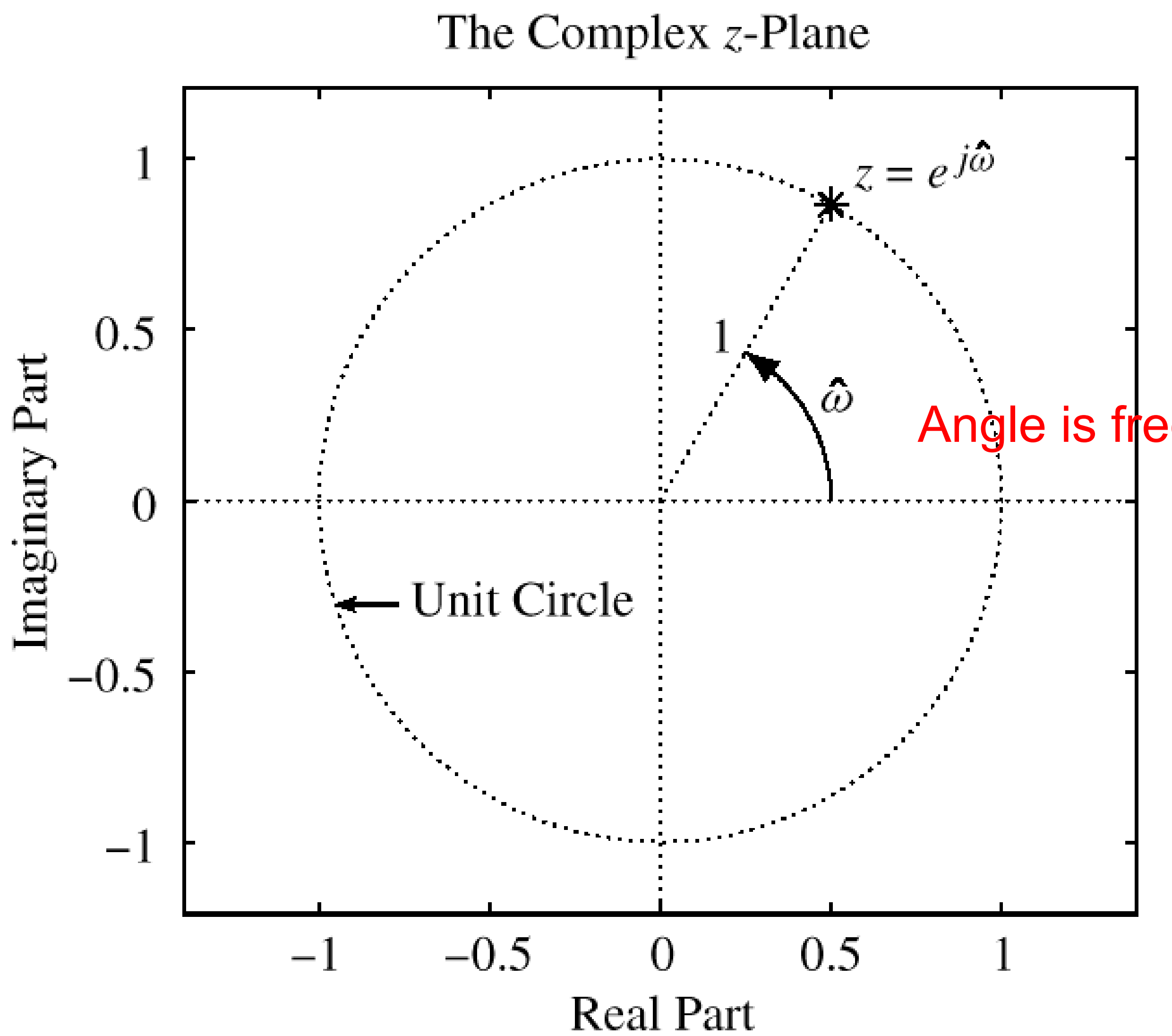
$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

- Relationship between the z-Domain and the $\hat{\omega}$ Domain

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

Relationship Bet. the z-Domain and the $\hat{\omega}$ Domain (2/2)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$



Angle is frequency

The Zeros and Poles of *H*(*z*)

- Zeros: Values of z for which H(z) is zero.
- Poles: Values of z for which H(z) is undefined (infinite).

- Example $H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$

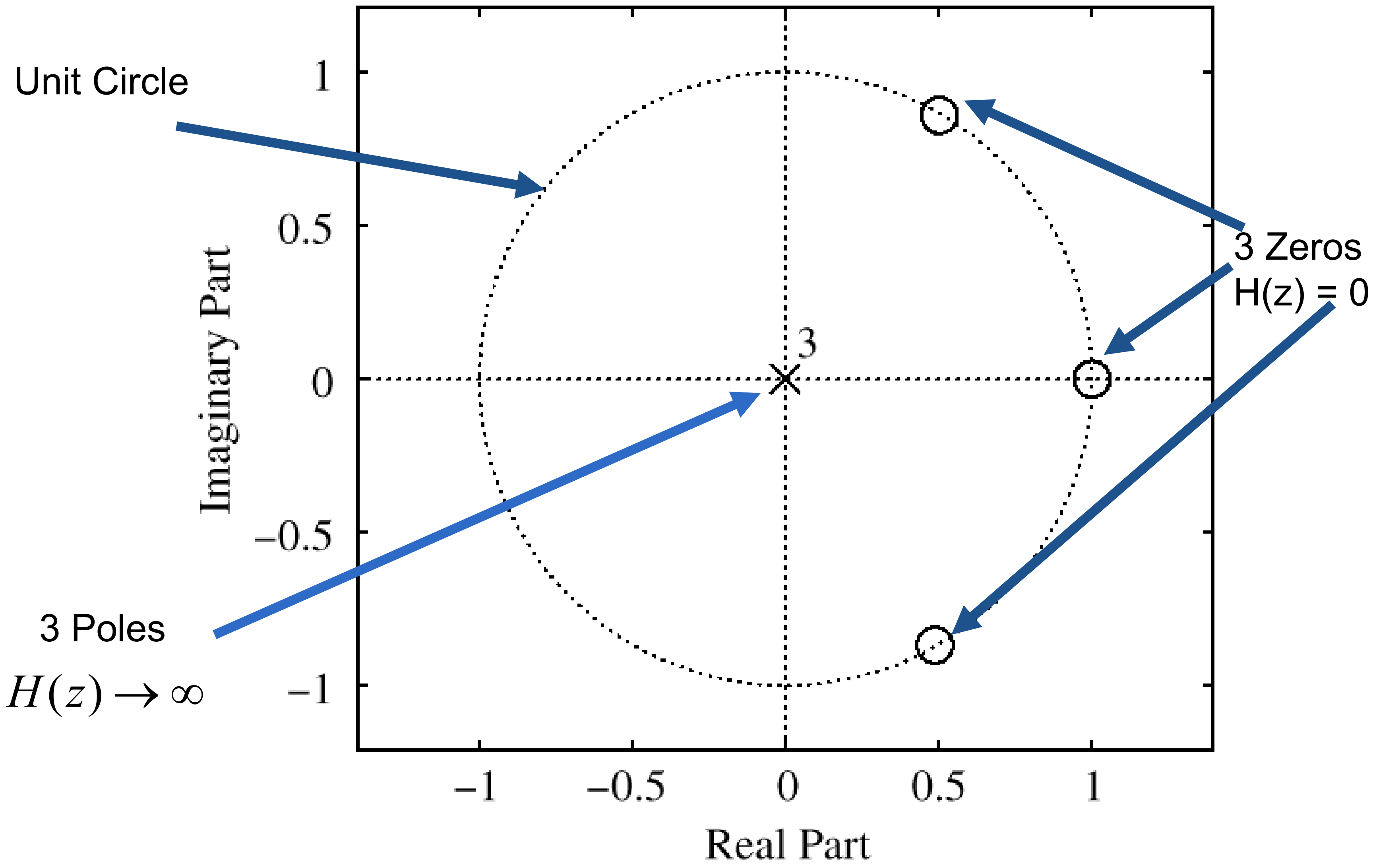
  – Zeros: $H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$

$$\text{zeros}: z = 1, \underline{\tfrac{1}{2} \pm j\tfrac{\sqrt{3}}{2}} \quad e^{\pm j\pi/3}$$

  – Poles: $H(z) = \dfrac{z^3 - 2z^2 + 2z - 1}{z^3}$

$$\text{Three Poles at}: z = 0$$

Pole-zero Plot



Nulling Filter

- If the zeros of H(z) lie on the unit circle, then certain sinusoidal input signals are removed or nulled by the filter.

- If we want to eliminate a sinusoidal input signal,

$$x[n] = \cos(\hat{\omega}_0 n) = \frac{1}{2} e^{j\hat{\omega}_0 n} + \frac{1}{2} e^{-j\hat{\omega}_0 n}$$

- For example, design a nulling filter to remove $x[n] = \cos(\frac{\pi}{4} n)$

Example: Nulling Filter

- Nulling at $z_1 = e^{j\hat{\omega}_0 n}$ and $z_2 = e^{-j\hat{\omega}_0 n}$

$$H_1(z) = 1 - z_1 z^{-1} \rightarrow H_1(z_1) = 1 - z_1 z_1^{-1} = 0$$

$$H_2(z) = 1 - z_2 z^{-1}$$

$$\begin{aligned} H(z) &= H_1(z) H_2(z) \\ &= (1 - z_1 z^{-1})(1 - z_2 z^{-1}) \\ &= 1 - (z_1 + z_2) z^{-1} + (z_1 z_2) z^{-2} \\ &= 1 - (e^{j\hat{\omega}_0 n} + e^{-j\hat{\omega}_0 n}) z^{-1} + (e^{j\hat{\omega}_0 n} e^{-j\hat{\omega}_0 n}) z^{-2} \\ &= 1 - 2\cos(\hat{\omega}_0) z^{-1} + z^{-2} \end{aligned}$$

$$H(z) = 1 - \sqrt{2} z^{-1} + z^{-2} \quad \longleftarrow \quad \hat{\omega}_0 = \pi / 4$$

$$y(n) = x[n] - \sqrt{2} x[n-1] + x[n-2]$$

L-pt Running Sum (1/2)

- System Function

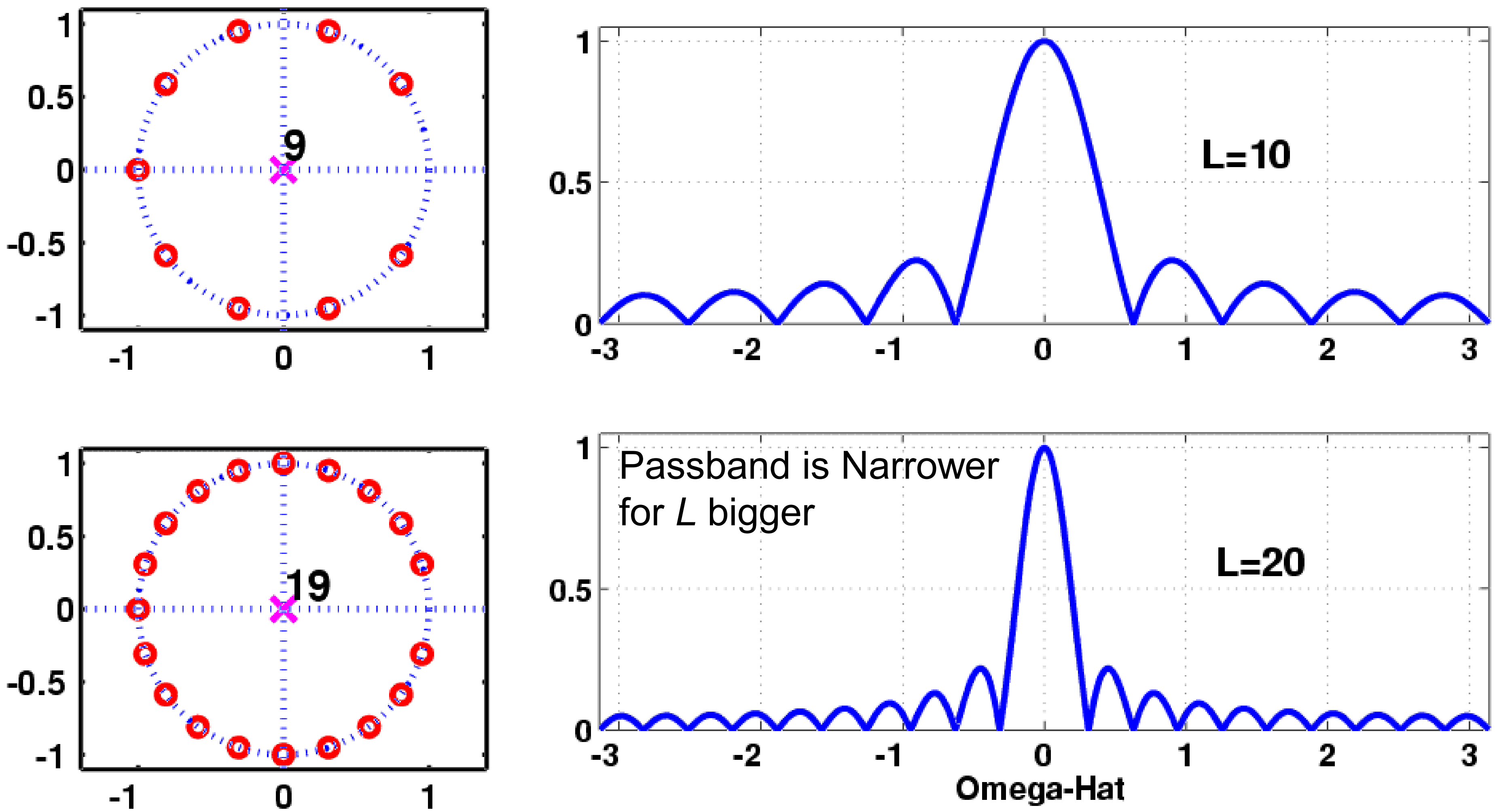
$$H(z) = \sum_{k=0}^{L-1} \frac{1}{L} z^{-k} = \frac{1 - z^{-L}}{L(1 - z^{-1})} = \frac{z^L - 1}{Lz^{L-1}(z-1)}$$

- Zeros $\quad z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}$

$$z = e^{j(2\pi/L)k} \quad \text{for } k = 1, 2, \ldots L-1$$

- Poles $\quad z^{L-1} = 0 \quad \Rightarrow \quad z = 0$

L-pt Running Sum (2/2)



Summary

- This lecture introduced the z-transform method.
- The z-transform reduces the manipulation of LTI systems into simple operations on polynomials and rational functions.
- There are three domains:
  - the n-domain or time domain $$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$
  - the $\hat{\omega}$ -domain or frequency domain $$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$
  - the z-domain. $$H(z) = \sum_{n} h[n] z^{-n}$$