신호 및 시스템

V. IIR 필터

학습에 앞서

- 학습목표
  - IIR 필터의 동작을 이해한다.
  - IIR 필터의 pole, zero 개념을 이해한다.
  - IIR 필터의 주파수 응답을 학습한다.

The General IIR Difference Equation

- The general difference equation of digital filters is

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

  - $\{a_l\}$: feedback coefficients
  - $\{b_k\}$: feedforward coefficients
  - $N+M+1$: Number of coefficients
  - If $\{a_l\}$ are all zero, the difference equation reduces to the difference equation of an FIR system.

One Feedback Term

- First-order case where M = N = 1, i.e.,

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

- Example

$$y[n] = 0.8y[n-1] + 5x[n]$$

$$y[0] = 0.8y[-1] + 5x[0]$$

Need y[-1] to get started

- Initial Rest Condition: *y*[n] = 0, for *n*<0 because *x*[n] = 0, for *n*<0

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

Time-Domain Response (1/2)

- Compute y[n]

$$y[n] = 0.8y[n-1] + 5x[n]$$

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$

- Continue the recursion

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

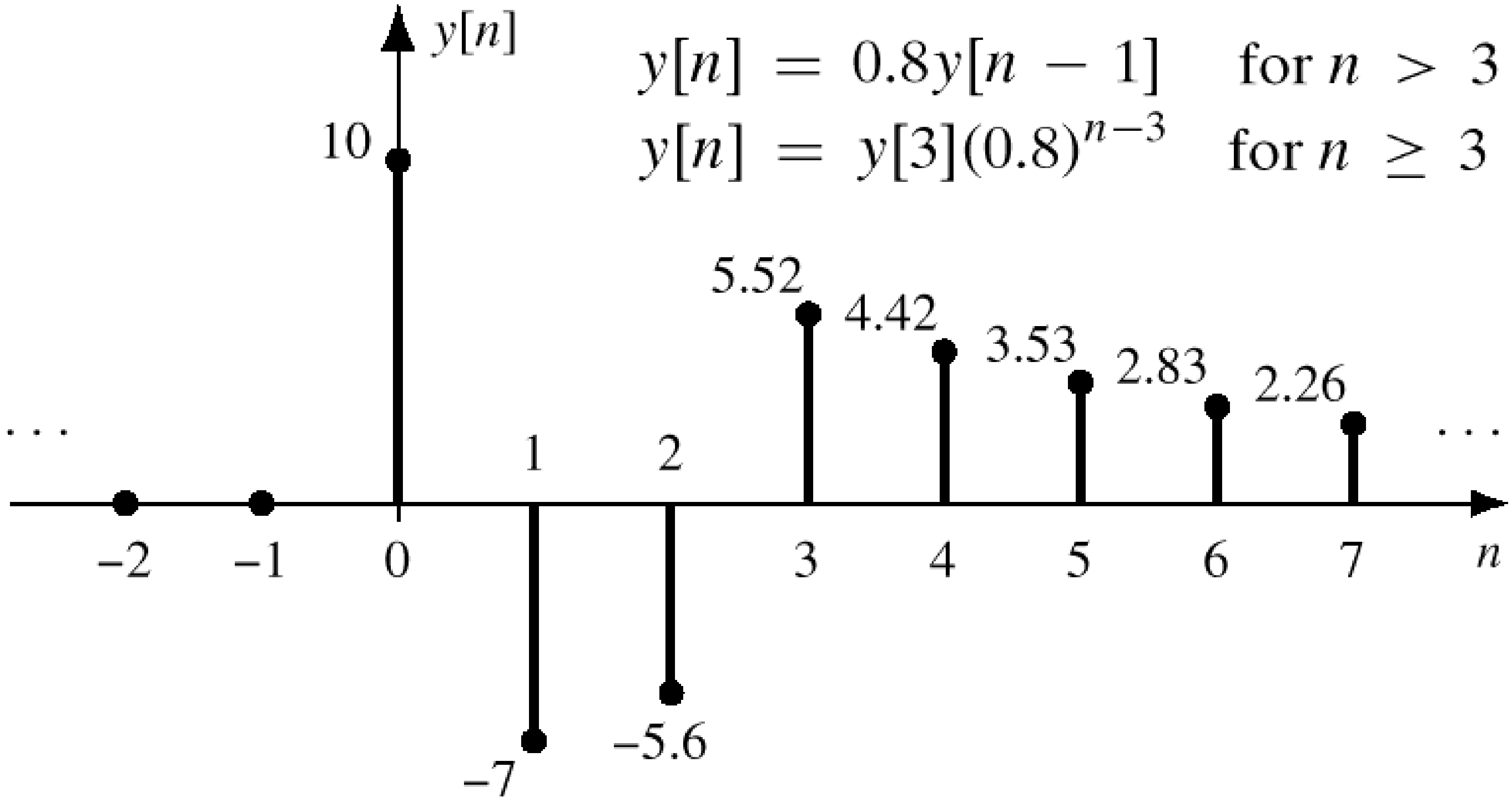
$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

Time-Domain Response (2/2)

- Plot y[n]



Impulse Response of a First-Order IIR System

- Consider the first-order recursive difference equation with b_1=0.

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

- Impulse Response

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

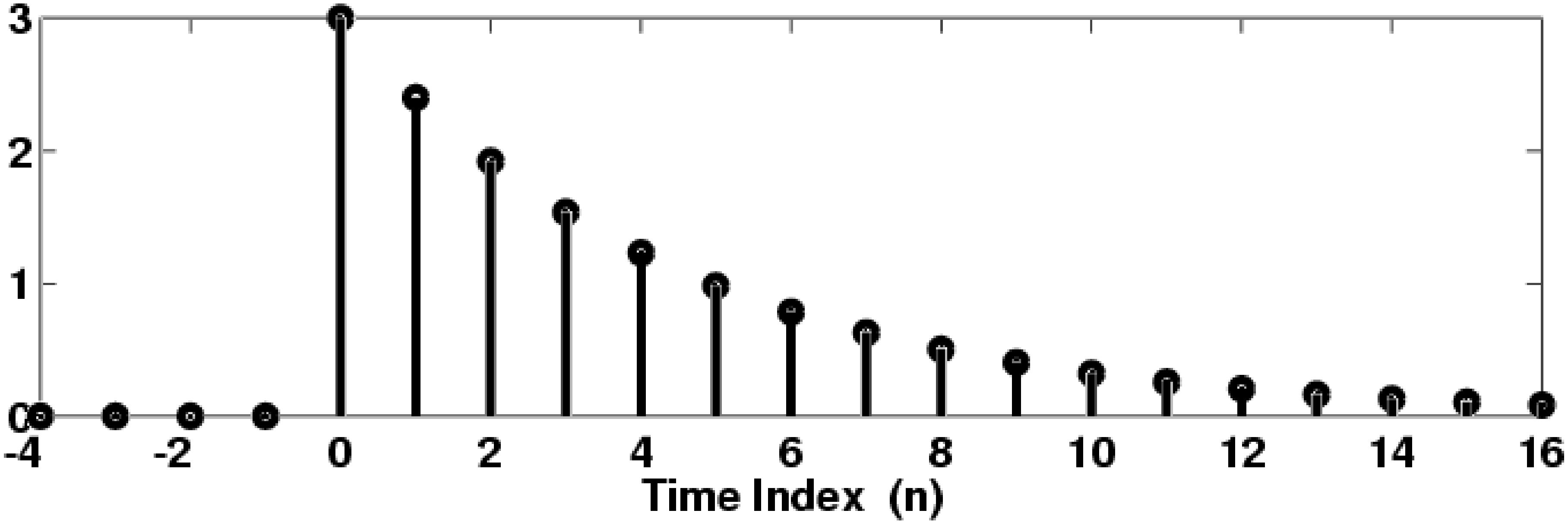
$$\boxed{h[n] = b_0(a_1)^n u[n]}$$

Example: Impulse Response

- Example

$$y[n] = 0.8y[n-1] + 3x[n]$$

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



System Function

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 a_1^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \boxed{\frac{b_0}{1 - a_1 z^{-1}}} \quad \text{if } |z| > |a_1|$$

Another First-Order IIR System

- Another First-Order IIR Filter

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

Because the system is linear and time-invariant, it follows

$$h[n] = b_0 (a_1)^n u[n] + \underline{b_1 (a_1)^{n-1} u[n-1]}$$

z^{-1} is a shift

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

Step Response of a First-Order Recursive System (1/2)

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

n	$x[n]$	$y[n]$
$n<0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
$\vdots$	1	$\vdots$

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

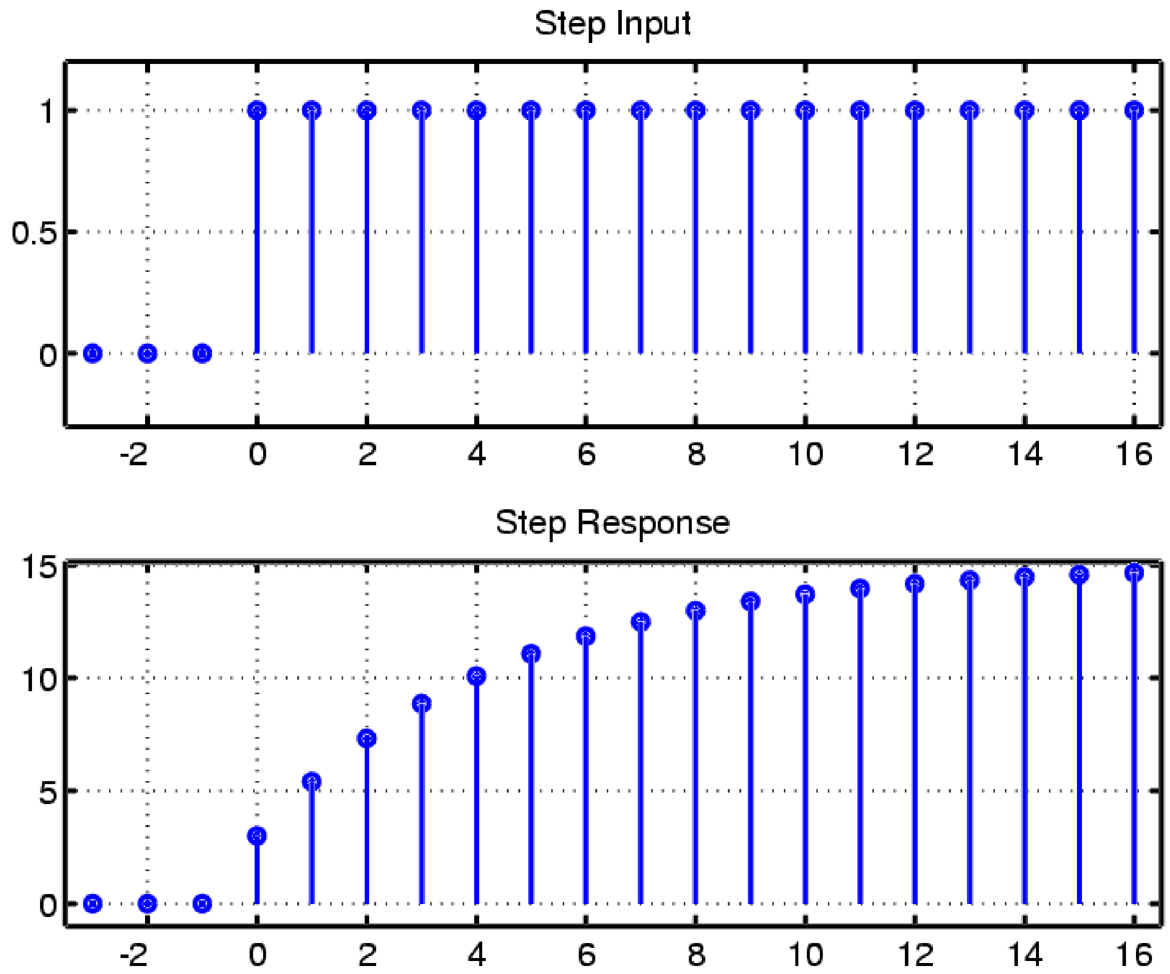
$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \quad \text{for } n \geq 0, \qquad \text{if } a_1 \neq 1$$

Step Response of a First-Order Recursive System (2/2)

- Plot Step Response

$$y[n] = 0.8y[n-1] + 3u[n]$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} u[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$$



Delay Property

- Delay in Time-Domain <-->Multiply X(z) by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$\boxed{x[n-1] \leftrightarrow z^{-1}X(z)}$$

Proof: $$\sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+1)}$$

$$= z^{-1}\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} = z^{-1}X(z)$$

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0}X(z)$$

System Function of an IIR Filter

- System function of the first-order difference equation

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

- Example

$$y[n] = 0.8 y[n-1] + 2x[n] + 2x[n-1]$$

$$Y(z) = \left(\frac{2 + 2z^{-1}}{1 - 0.8 z^{-1}} \right) X(z)$$

Poles and Zeros (1/2)

- Roots of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

  - Zeros at H(z)=0

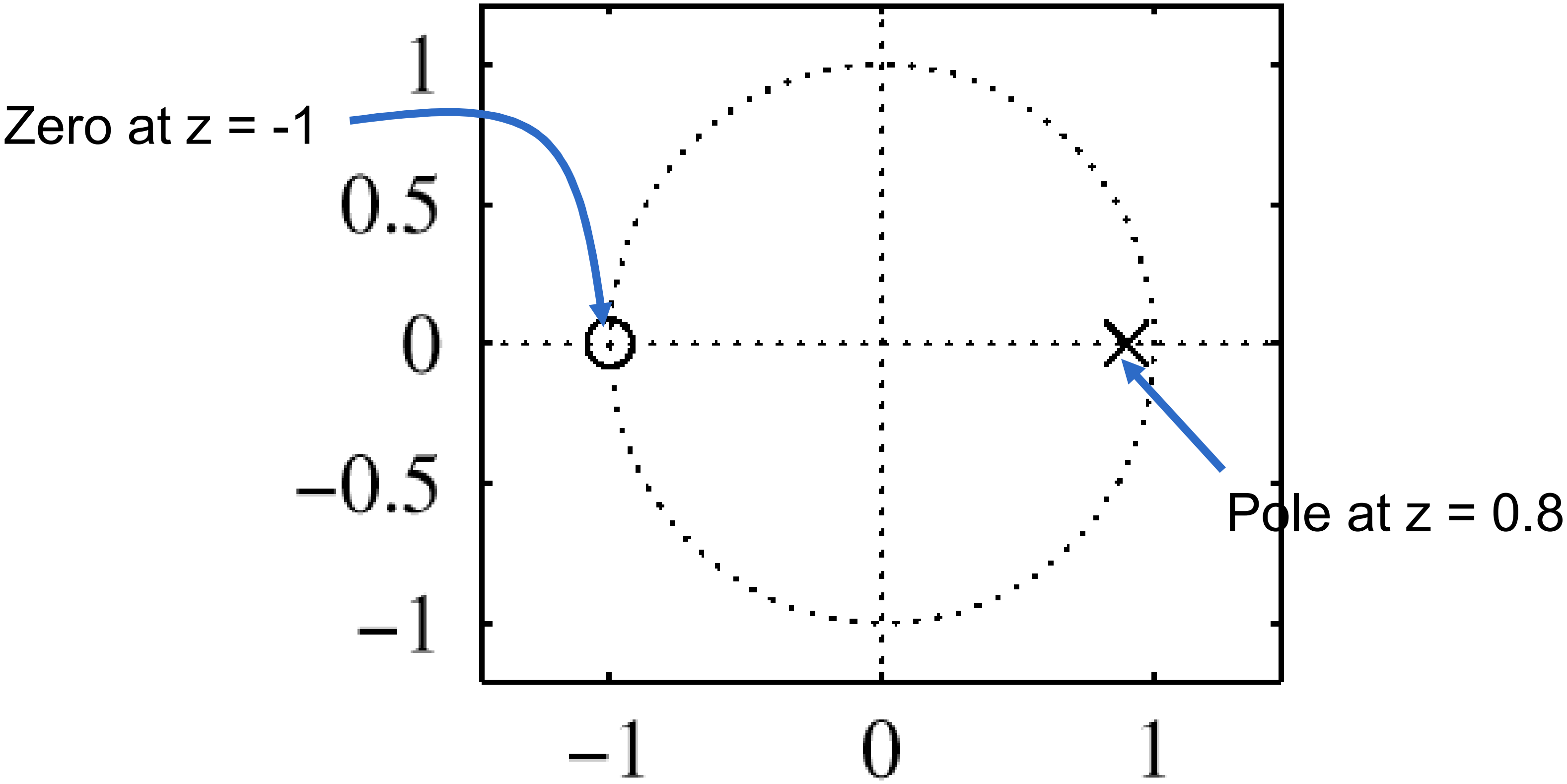
$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

  - Poles at H(z)→∞

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

Poles and Zeros (2/2)

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$



Frequency Response

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

- Example

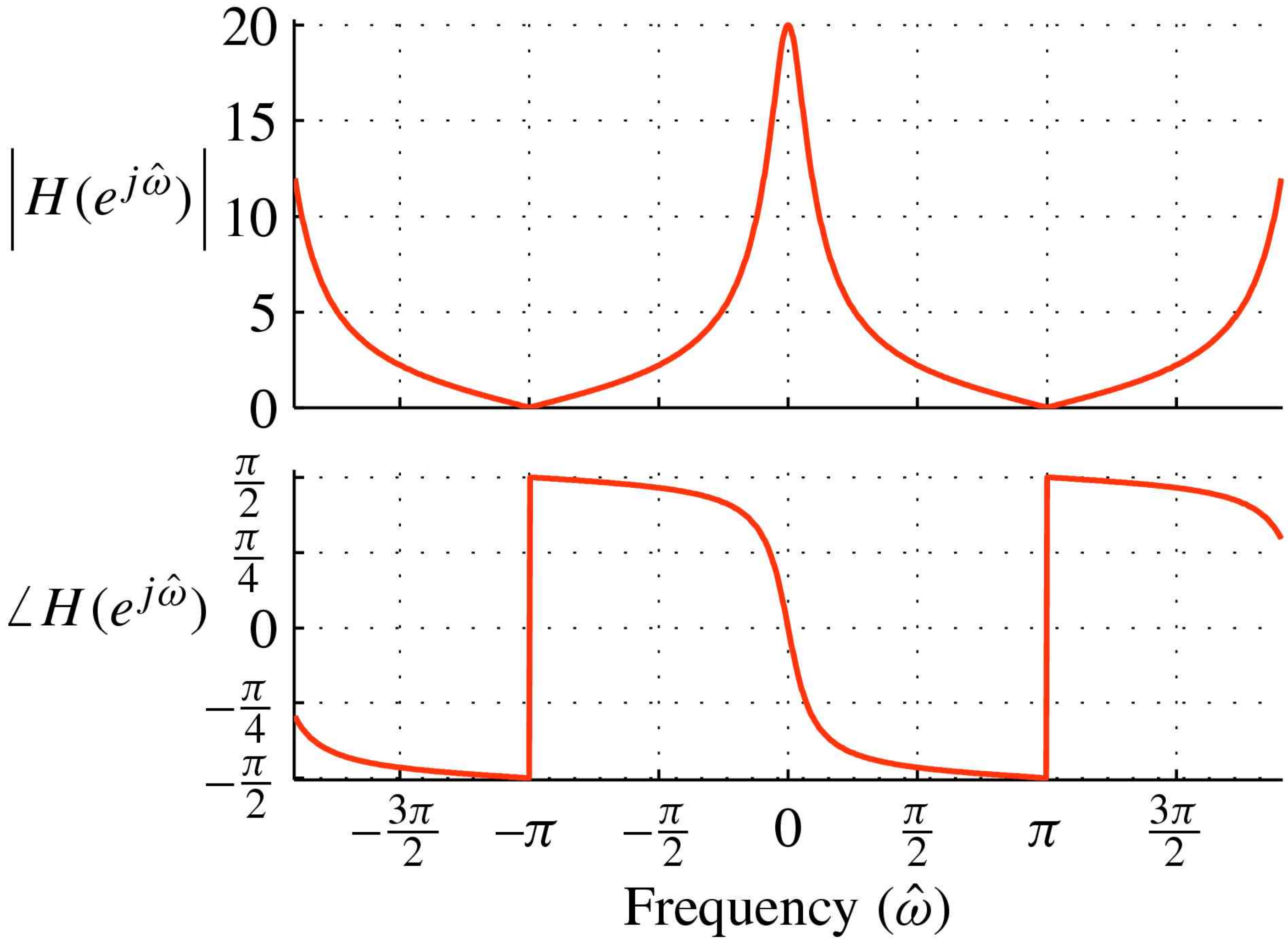
$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$

$$\left|H(e^{j\hat{\omega}})\right|^2 = H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}} \cdot \frac{2 + 2e^{j\hat{\omega}}}{1 - 0.8e^{j\hat{\omega}}} = \frac{8 + 8\cos\hat{\omega}}{1.64 - 1.6\cos\hat{\omega}}$$

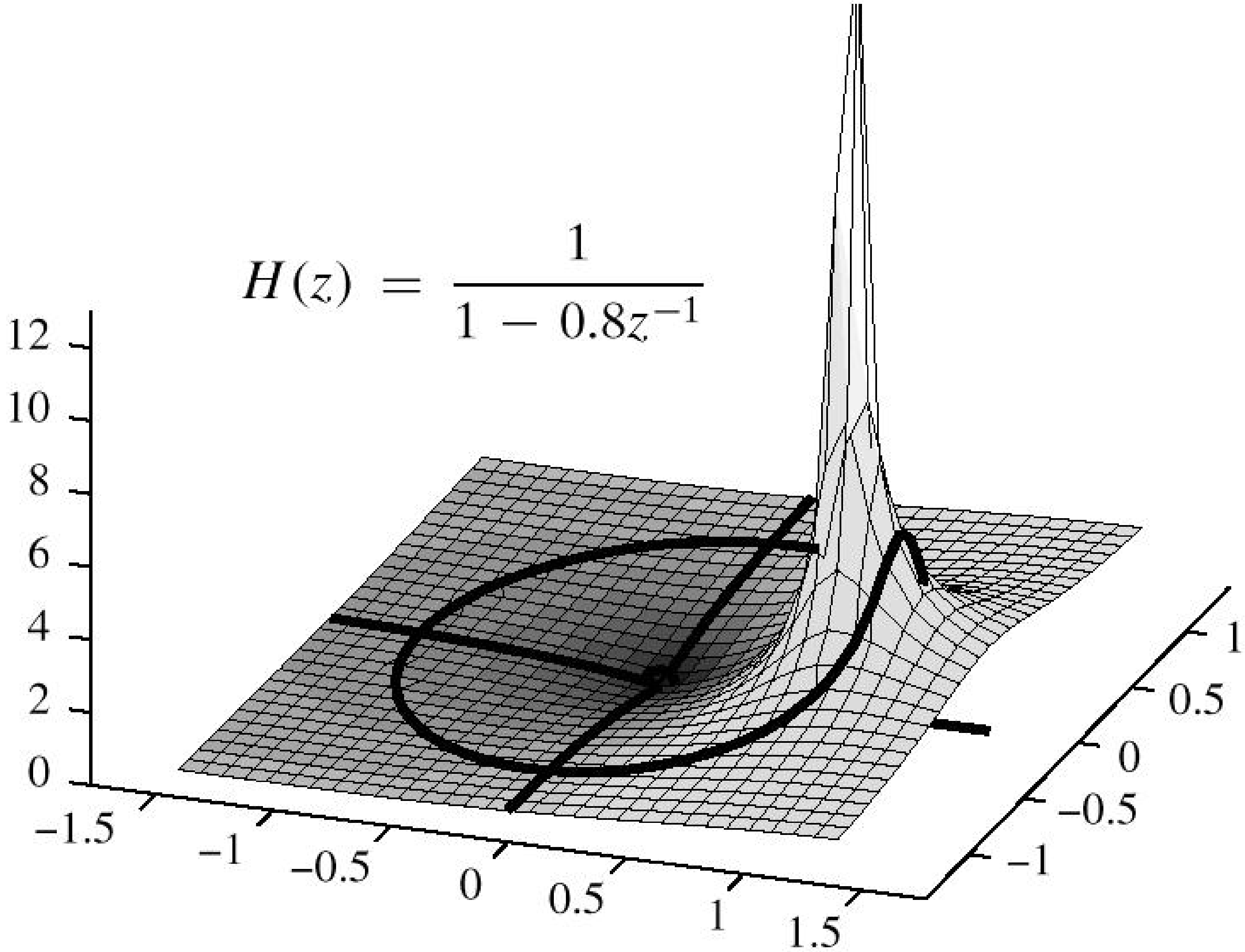
$$\left|H(e^{j\hat{\omega}})\right|^2 = \frac{8 + 8}{0.04} = 400 \quad @\,\hat{\omega} = 0$$

Frequency Response Plot

$$H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$

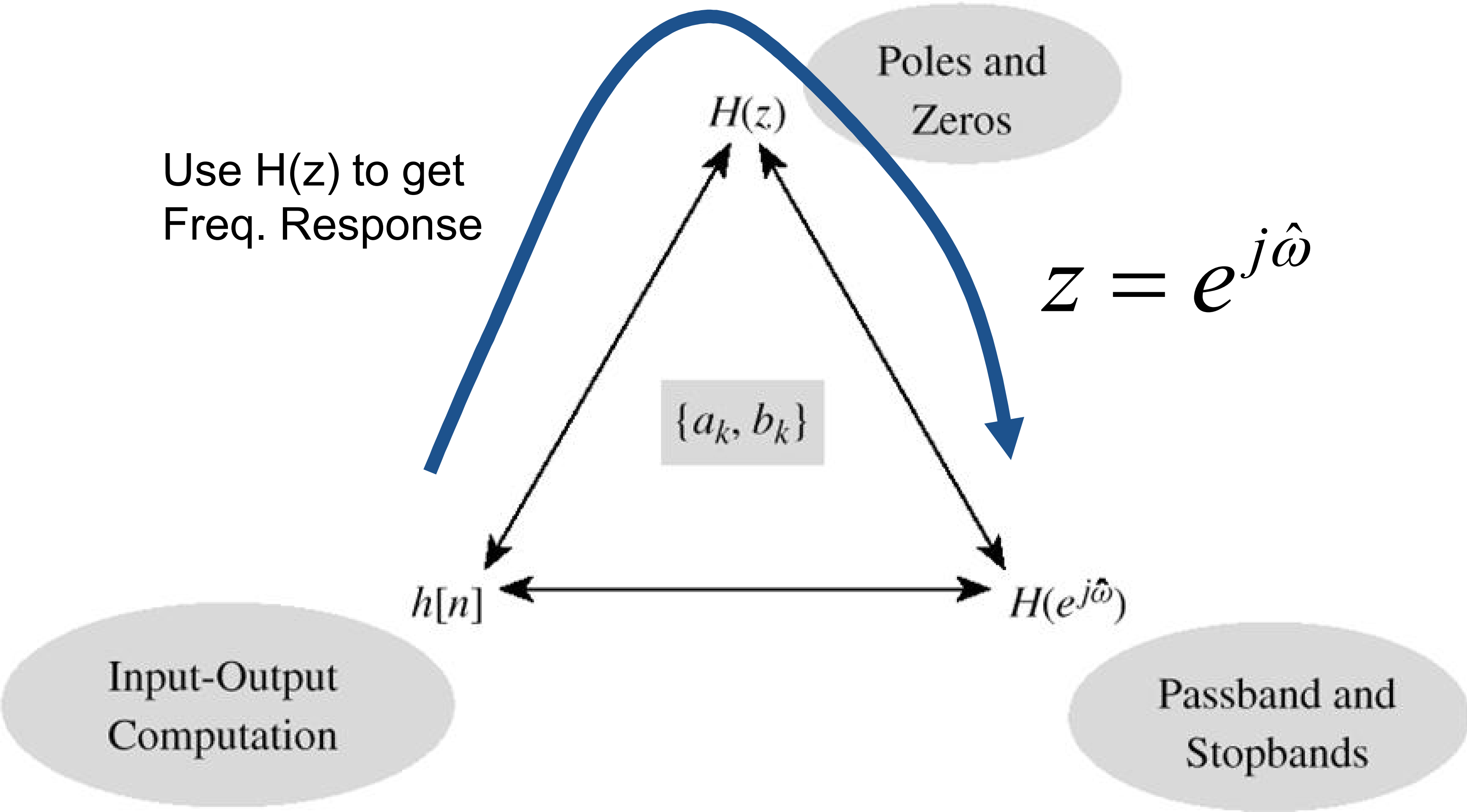


Three-Dimensional Plot of a System Function



Three Domains

- Relationship among the n-, z-, and $\hat{\omega}-$ domains.



z-Transform Tables

	SHORT TABLE OF z-TRANSFORMS		
	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1 - az^{-1}}$

Summary

- This lecture introduced a new class of LTI systems that have infinite duration impulse responses, i.e., a IIR system.
  - IIR digital filters involve previously computed values of the output signal as well as values of the input signal in the computation of the present output.

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

- The z-transform system functions for IIR filters are rational functions that have poles and zeros.
  - Poles of the system function H(z) are important because properties such as the shape of the frequency response or the form of the impulse response can be inferred quickly from the pole locations.