신호 및 시스템

VI. IIR 필터(2), LTI 시스템

학습에 앞서

- 학습목표

  1) IIR 필터
  - 2차 IIR 필터를 학습한다.
  - H(z)로부터 h[n]를 구하는 방법을 학습한다.

  2) 연속 시간 LTI 시스템
  - 선형성과 시불변의 개념을 이해한다.
  - LTI 시스템의 컨벌루션 표현을 이해한다.

The General IIR Difference Equation

- The general difference equation of digital filters is

$$y[n] = \sum_{l=1}^{N} a_l y[n-l] + \sum_{k=0}^{M} b_k x[n-k]$$

  - $\{a_l\}$: feedback coefficients
  - $\{b_k\}$: feedforward coefficients
  - $N+M+1$: Number of coefficients
  - If $\{a_l\}$ are all zero, the difference equation reduces to the difference equation of an FIR system.

Second-Order IIR Filters

- Two feedback terms

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2} \rightarrow \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

Example (1/2)

$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n] = (0.9)^n \tfrac{1}{2}(e^{j\pi n/3} + e^{-j\pi n/3}) u[n]$$

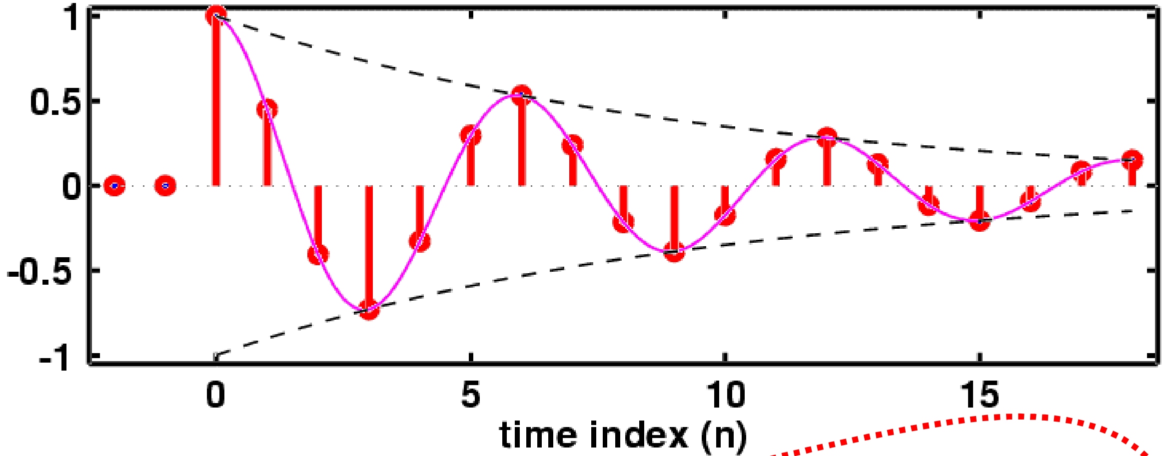
$$H(z) = \frac{0.5}{1 - 0.9e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9e^{-j\pi/3} z^{-1}}$$

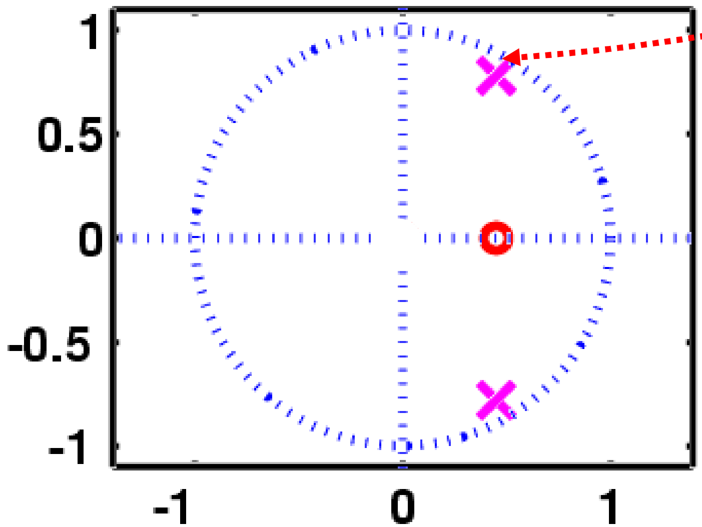
$$= \frac{1 - 0.9\cos(\tfrac{\pi}{3}) z^{-1}}{(1 - 0.9e^{j\pi/3} z^{-1})(1 - 0.9e^{-j\pi/3} z^{-1})}$$

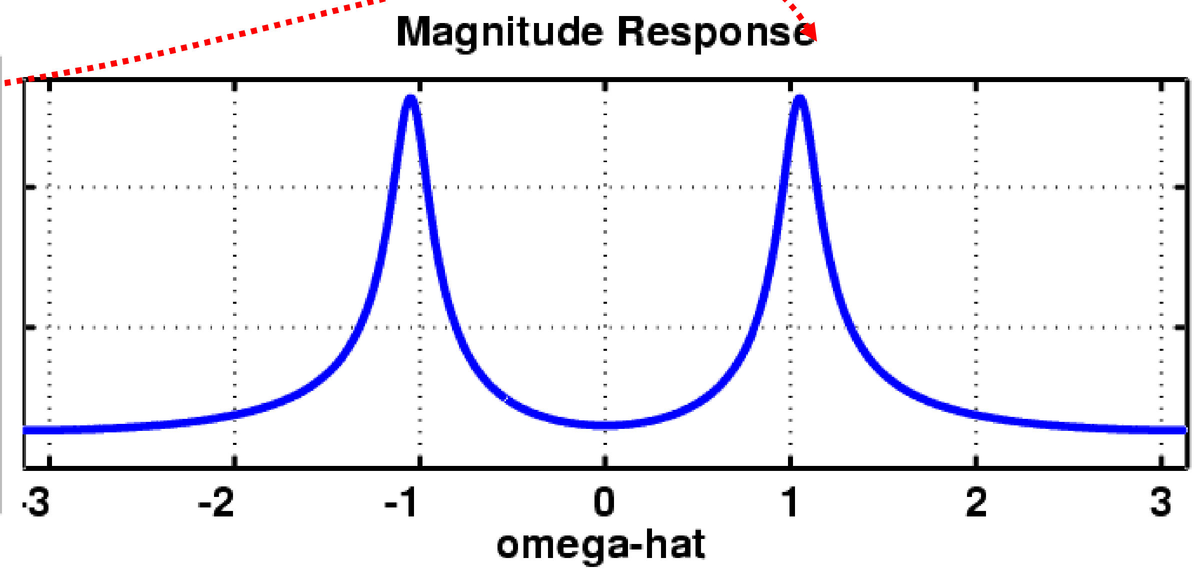
$$= \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

Example (2/2)

$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n] \qquad H(z) = \frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$







A General Procedure for Inverse z-Transformation (1/2)

- Partial fraction expansion

$$Y(z) = H(z)X(z) = \frac{b_0 + b_1 z^{-1}}{(1 - a_1 z^{-1})(1 - z^{-1})} = \frac{A}{1 - a_1 z^{-1}} + \frac{B}{1 - z^{-1}}$$

$$Y(z)(1 - a_1 z^{-1}) = \frac{b_0 + b_1 z^{-1}}{(1 - z^{-1})} = A + \frac{B(1 - a_1 z^{-1})}{1 - z^{-1}}$$

$$A = Y(z)(1 - a_1 z^{-1})\Big|_{z=a_1} = \frac{b_0 + b_1 a_1^{-1}}{1 - a_1^{-1}}$$

$$B = Y(z)(1 - z^{-1})\Big|_{z=1} = \frac{b_0 + b_1}{1 - a_1}$$

$$y[n] = A\, a_1^n u[n] + B\, u[n]$$

A General Procedure for Inverse z-Transformation (2/2)

PROCEDURE FOR INVERSE z-TRANSFORMATION $(M < N)$

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$.
2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1 - p_k z^{-1})\Big|_{z=p_k}$$

3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

Example 1: Inverse z-Transformation

- Partial fraction expansion

$$X(z) = \frac{1 - 2.1z^{-1}}{1 - 0.3z^{-1} - 0.4z^{-2}} = \frac{1 - 2.1z^{-1}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})}$$

$$X(z) = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.8z^{-1}}$$

$$A = X(z)(1 + 0.5z^{-1})\big|_{z=-0.5} = \left.\frac{1 - 2.1z^{-1}}{1 - 0.8z^{-1}}\right|_{z=-0.5} = \frac{1 + 4.2}{1 + 1.6} = 2$$

$$B = X(z)(1 - 0.8z^{-1})\big|_{z=0.8} = \left.\frac{1 - 2.1z^{-1}}{1 + 0.5z^{-1}}\right|_{z=0.8} = \frac{1 - 2.1/0.8}{1 + 0.5/0.8} = -1$$

$$x[n] = 2(-0.5)^n u[n] - (0.8)^n u[n]$$

Example 2: Inverse z-Transformation

- Long Division Case

$$Y(z) = \frac{2 - 2.4z^{-1} - 0.4z^{-2}}{1 - 0.3z^{-1} - 0.4z^{-2}} = \frac{2 - 2.4z^{-1} - 0.4z^{-2}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})}$$

$$Y(z) = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.8z^{-1}} + C$$

$$\begin{array}{r|l} & \quad 1 \\ \hline -0.4z^{-2} - 0.3z^{-1} + 1 & -0.4z^{-2} - 2.4z^{-1} + 2 \\ & -0.4z^{-2} - 0.3z^{-1} + 1 \\ \hline & \qquad\quad - 2.1z^{-1} + 1 \end{array}$$

$$Y(z) = \frac{1 - 2.1z^{-1}}{(1 + 0.5z^{-1})(1 - 0.8z^{-1})} + 1$$

$$Y(z) = \frac{2}{1 + 0.5z^{-1}} - \frac{1}{1 - 0.8z^{-1}} + 1$$

$$y[n] = 2(-0.5)^n u[n] - (0.8)^n u[n] + \delta[n]$$

Steady-State Response and Stability (1/2)

- Consider LTI system defined by $y[n] = a_1 y[n-1] + b_0 x[n]$

- The system function $H(z) = \dfrac{b_0}{1 - a_1 z^{-1}}$

- Impulse response $h[n] = b_0 a_1^n u[n]$

- Frequency response $H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \dfrac{b_0}{1 - a_1 e^{-j\hat{\omega}}}$

- The output for a complex exponential input is

$$y[n] = H(e^{j\hat{\omega}_0})e^{j\hat{\omega}_0 n} = \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n}$$

Steady-State Response and Stability (2/2)

- What if the complex exponential input sequence is suddenly applied instead of existing for all n?
- For the suddenly applied complex exponential sequence with frequency $\hat{\omega}_0$

$$x[n] = e^{j\hat{\omega}_0 n}u[n] \qquad X(z) = \frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}$$

- z-transform of the output

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}\right)$$

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n]$$

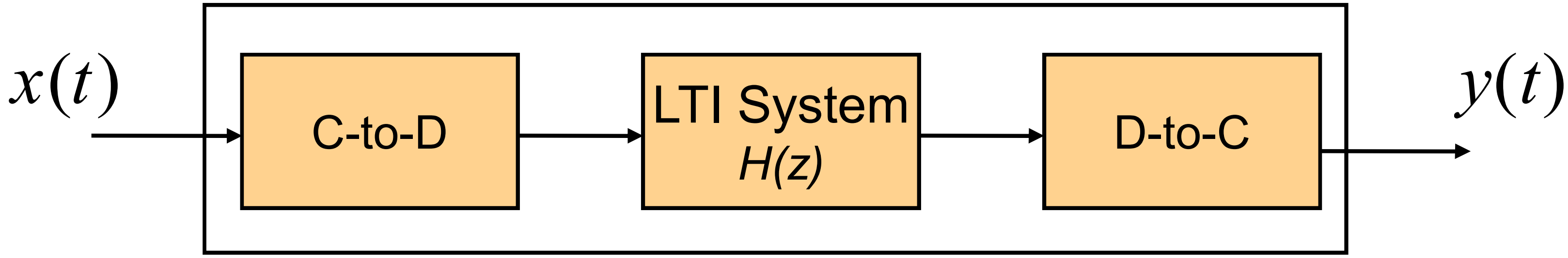
Transient component

$$+ \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)e^{j\hat{\omega}_0 n}u[n]$$

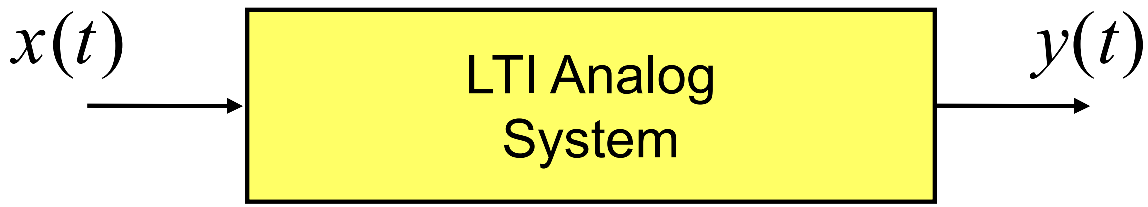
Sinusoidal steady-state component

Continuous-Time Signals and LTI Systems

- D-T Filtering of C-T Signals

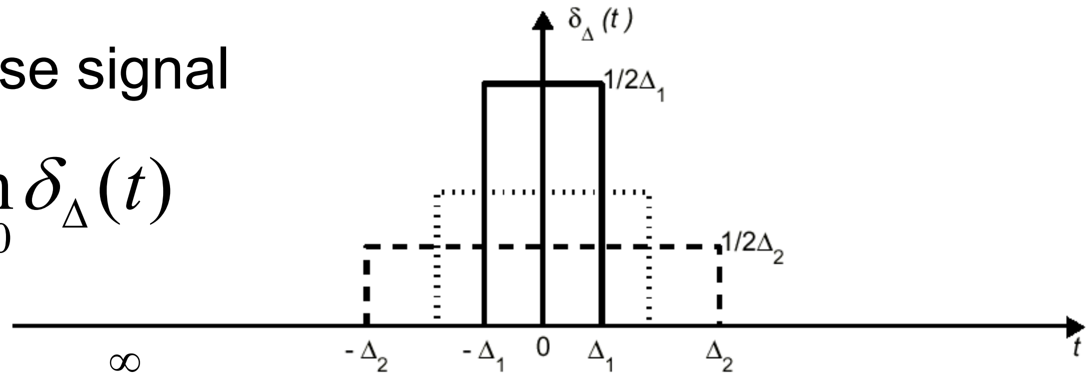


$$\hat{\omega} = \omega T_s \quad \text{or} \quad \omega = \hat{\omega} f_s$$



The Unit Impulse

- Definition of the unit impulse signal



$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

- Properties
  - Unit area

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$$

  - Concentrated at one time

$$\delta(t - t_0) = 0, \quad t \neq t_0$$

  - Sampling Property

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$$

  - Extract one value of f(t)

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0) dt = f(t_0)$$

  - Derivative of unit step

$$\frac{du(t)}{dt} = \delta(t)$$

The Convolution Integral

- If a continuous-time system is both linear and time-invariant, then the output y(t) is related to the input x(t) by a convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- Properties of Convolution
  - Linear $\quad x(t) = ax_1(t) + bx_2(t) \rightarrow y(t) = ay_1(t) + by_2(t)$
  - Commutative $\quad y(t) = h(t) * x(t) = x(t) * h(t)$
  - Time-Invariant $\quad x(t-t_0) \rightarrow y(t-t_0)$

Stability

- A system is stable if every bounded input produces a bounded output.
- A continuous-time LTI system is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

- Proof

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \right| \leq \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)| d\tau$$

$$\leq B \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

By Cauchy–Schwarz inequality

Causality

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$.
- An LTI system is causal if and only if

$$h(t) = 0 \ \text{ for } \ t < 0$$

- Proof

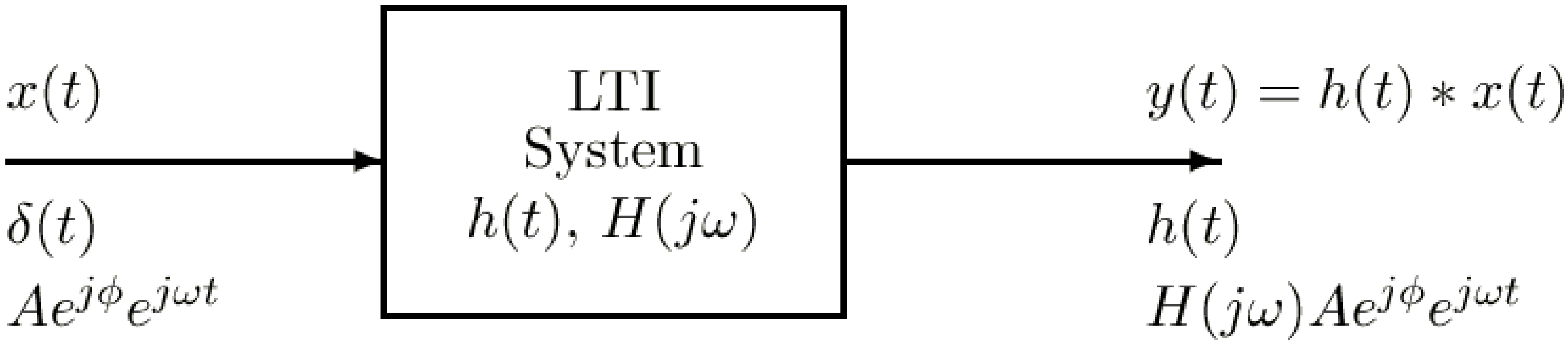
$$y(t_0) = \int_{-\infty}^{\infty} x(\tau) h(t_0 - \tau) d\tau$$

$$h(t_0 - \tau) = 0 \ \text{ for } \ \tau > t_0 \ \text{ from } \ h(\tau) = 0 \ \text{ for } \ \tau < 0$$

$$y(t_0) = \int_{-\infty}^{t_0} x(\tau) h(t_0 - \tau) d\tau$$

$$y(t_0) \ \text{ depends only on values of the input } \ x(\tau) \ \text{ for } \ \tau \leq t_0$$

LTI Systems



- Convolution defines an LTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Response to a complex exponential gives frequency response $H(j\omega)$

Frequency Response

- Complex Exponential Input

$$x(t) = Ae^{j\varphi}e^{j\omega t} \mapsto y(t) = H(j\omega)Ae^{j\varphi}e^{j\omega t}$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)Ae^{j\varphi}e^{j\omega(t-\tau)}d\tau$$

$$= \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau \right) Ae^{j\varphi}e^{j\omega t}$$

- Frequency Response of an LTI system

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

Example (1/2)

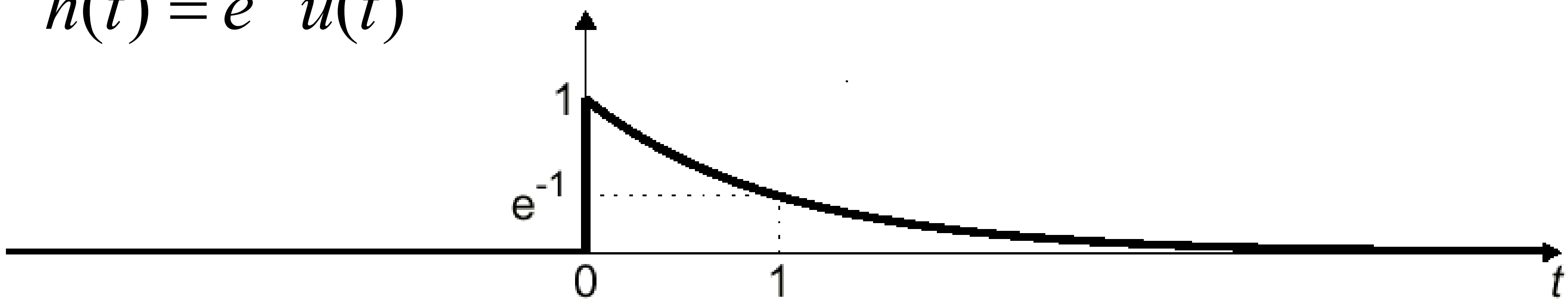
$$h(t)=e^{-at}u(t)\Leftrightarrow H(j\omega)=\frac{1}{a+j\omega}$$

- Proof

$$H(j\omega)=\int_{-\infty}^{\infty}e^{-a\tau}u(\tau)e^{-j\omega\tau}d\tau=\int_{0}^{\infty}e^{-(a+j\omega)\tau}d\tau$$

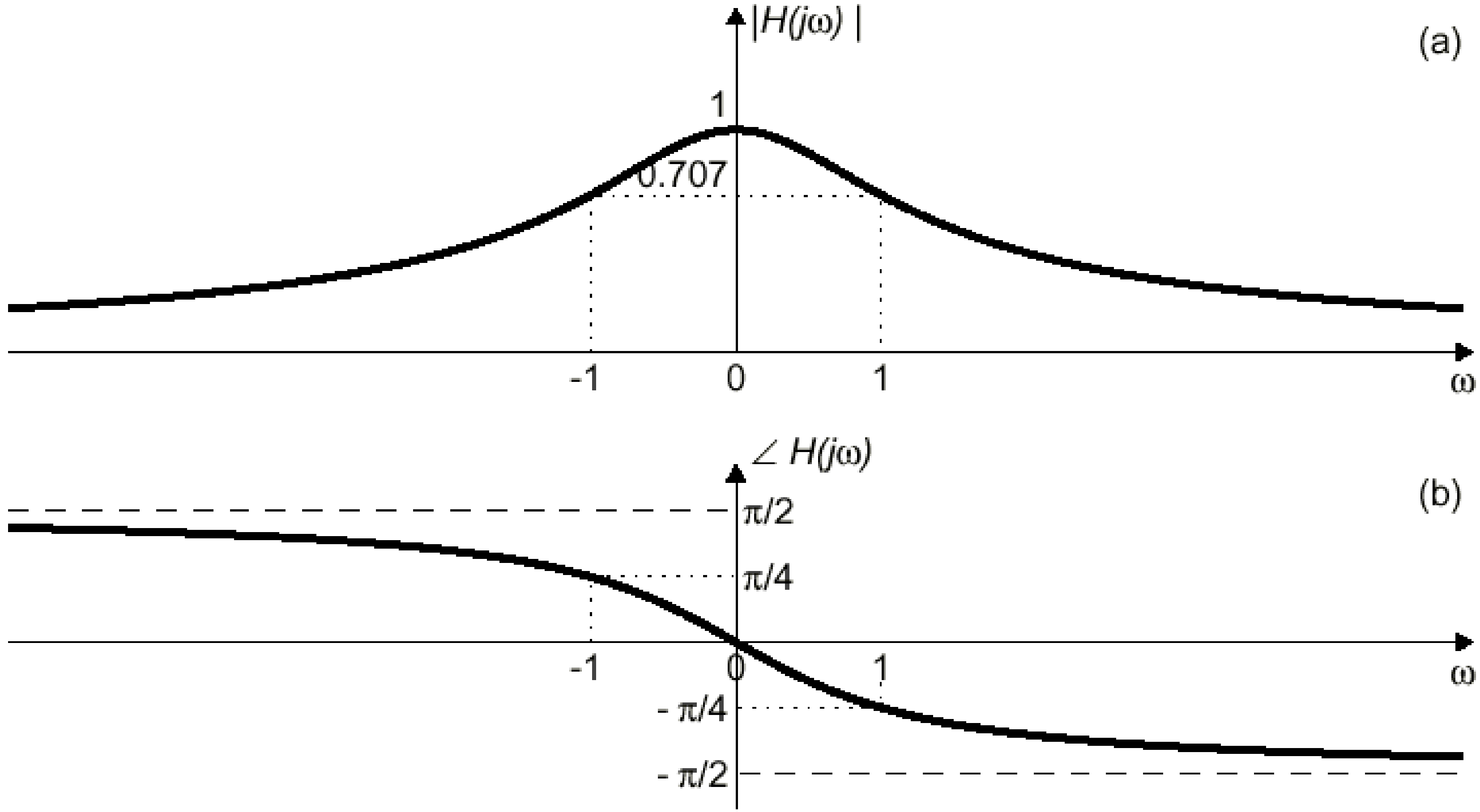
$$=\left.\frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)}\right|_0^{\infty}=\left.\frac{e^{-a\tau}e^{-j\omega\tau}}{-(a+j\omega)}\right|_0^{\infty}=\frac{1}{a+j\omega}$$

$$h(t)=e^{-t}u(t)$$



Example (2/2)

$$H(j\omega)=\frac{1}{1+j\omega}\qquad\left|\frac{1}{1+j\omega}\right|=\left|\frac{1}{\sqrt{1+\omega^2}}\right|\qquad\angle H(j\omega)=-\mathrm{atan}(\omega)$$



Ideal Delay System

- The ideal delay system is defined by the input/output relation

$$y(t) = x(t - t_d)$$

- Frequency response of the ideal delay system

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - t_d) e^{-j\omega\tau} d\tau = e^{-j\omega t_d}$$

- Example

$$x(t) = e^{j\omega t} \mapsto$$

$$y(t) = e^{j\omega(t - t_d)} = \left(e^{-j\omega t_d}\right) e^{j\omega t}$$

Summary

- Second-order IIR filters $y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$
  - Determine h[n] from H(z) by using the Inverse z-Transformation.

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \Rightarrow h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

- Continuous-Time Signals and Systems
  - Linear
  - Time-Invariant
  - Convolution $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = x(t) * h(t)$
  - Stability: every bounded input produces a bounded output.
  - Causality: $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$.