신호 및 시스템

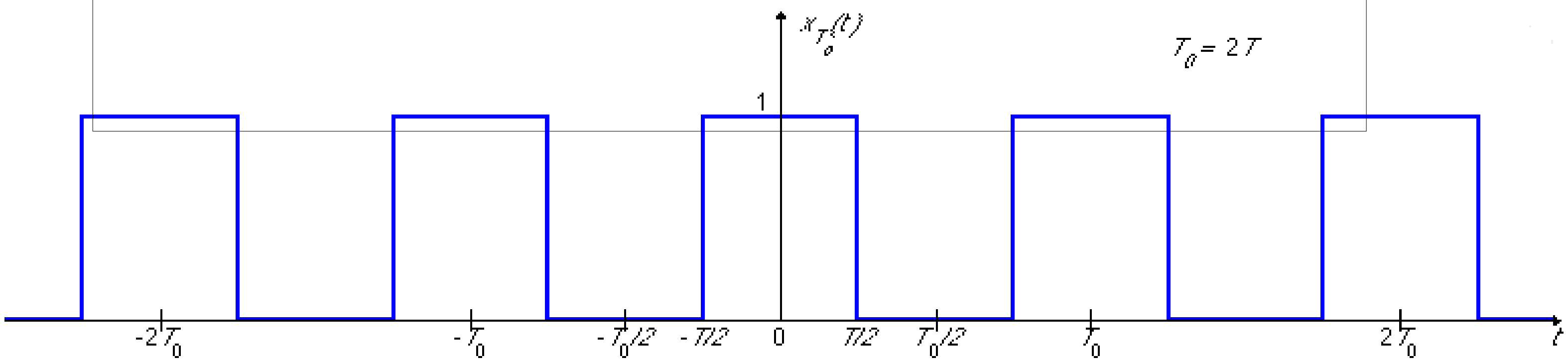
VII. Fourier Transform

학습에 앞서

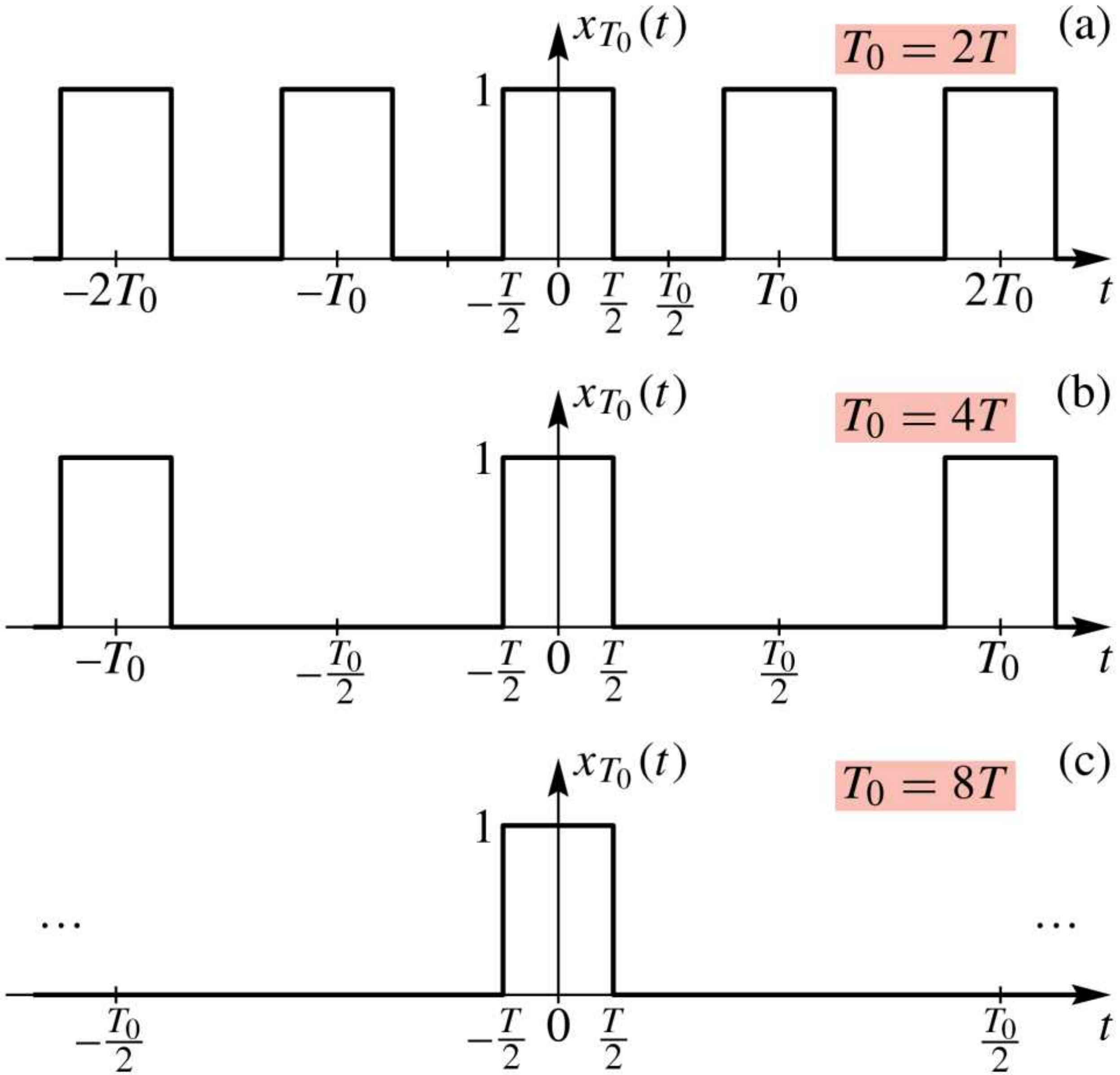
- 학습목표
  - 푸리에 변환의 정의를 이해한다.
  - 다양한 함수들의 푸리에 변환을 구해보고, 푸리에 변환의 특성을 학습한다.

Fourier Series (Periodic Signal)

- Everything = Sum of Sinusoids
- Fourier Series: Periodic of x(t) $x(t) = x(t + T_0)$
  - Fourier Synthesis $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$
  - Fourier Analysis $$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$$
  - Fundamental Frequency $$\omega_0 = 2\pi / T_0 = 2\pi f_0$$



Limit of the Fourier Series (Non-Periodic Signal)



- Fourier Synthesis

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left(T_0 a_k\right) e^{j\omega_0 kt} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\lim_{T_0 \to \infty} T_0 a_k = X(j\omega) \qquad \lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega \qquad \lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega$$

- Fourier Analysis

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 kt} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform

- Fourier Synthesis (Inverse Transform)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

- Fourier Analysis (Forward Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- Time-Domain $\Leftrightarrow$ Frequency-Domain

$$x(t) \Leftrightarrow X(j\omega)$$

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Examples of Fourier Transform Pairs (1/6)

(1) Right-Sided Real Exponential Signals

$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega}$$

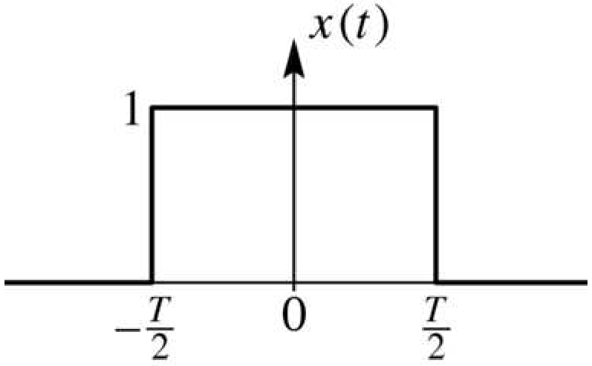
$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = \int_0^\infty e^{-(a+j\omega)t} dt$$

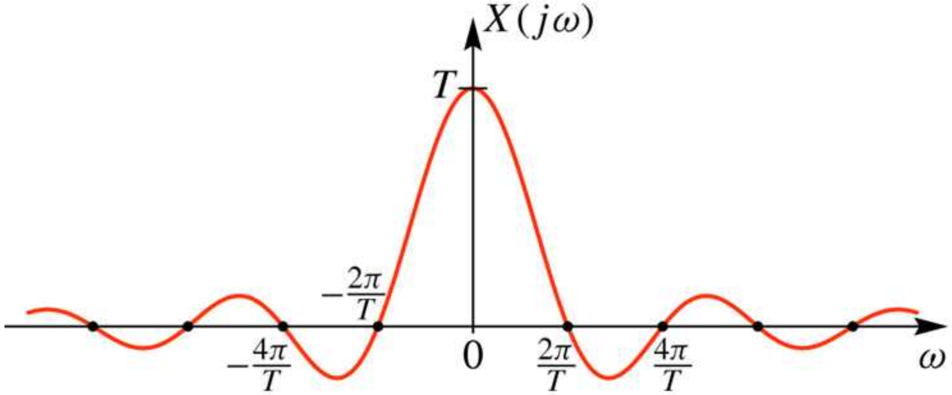
$$= -\left.\frac{e^{-at} e^{-j\omega t}}{a + j\omega}\right|_0^\infty = \frac{1}{a + j\omega}$$

- Fourier Transform of *h*(t) is the Frequency Response

(2) Rectangular Pulse Signals

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \quad \Leftrightarrow \quad X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$





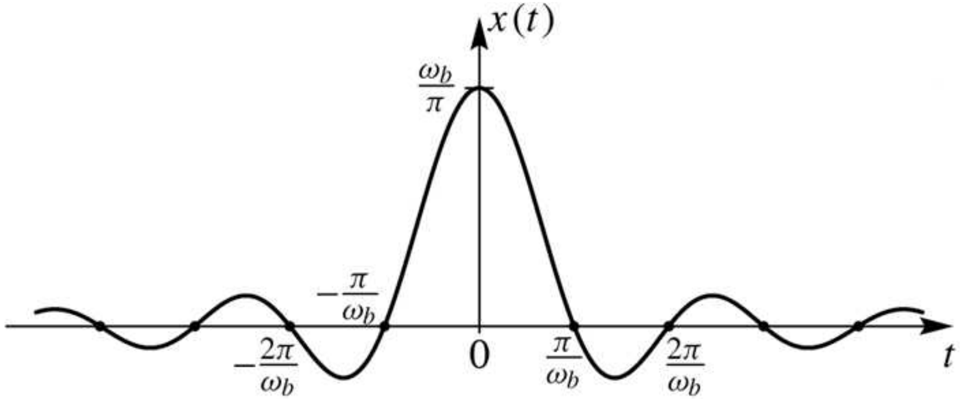
$$X(j\omega)=\int_{-T/2}^{T/2}(1)e^{-j\omega t}dt=\int_{-T/2}^{T/2}e^{-j\omega t}dt$$

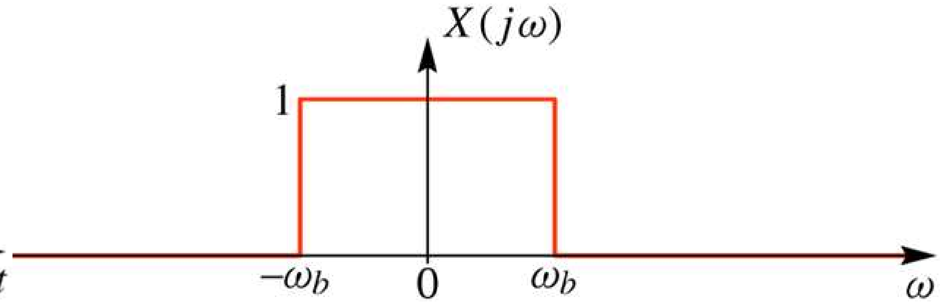
$$=\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T/2}^{T/2}=\frac{e^{-j\omega T/2}-e^{j\omega T/2}}{-j\omega}=\frac{\sin(\omega T/2)}{(\omega/2)}$$

Examples of Fourier Transform Pairs (3/6)

(3) Bandlimited Signals (Low Pass Filter)

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$





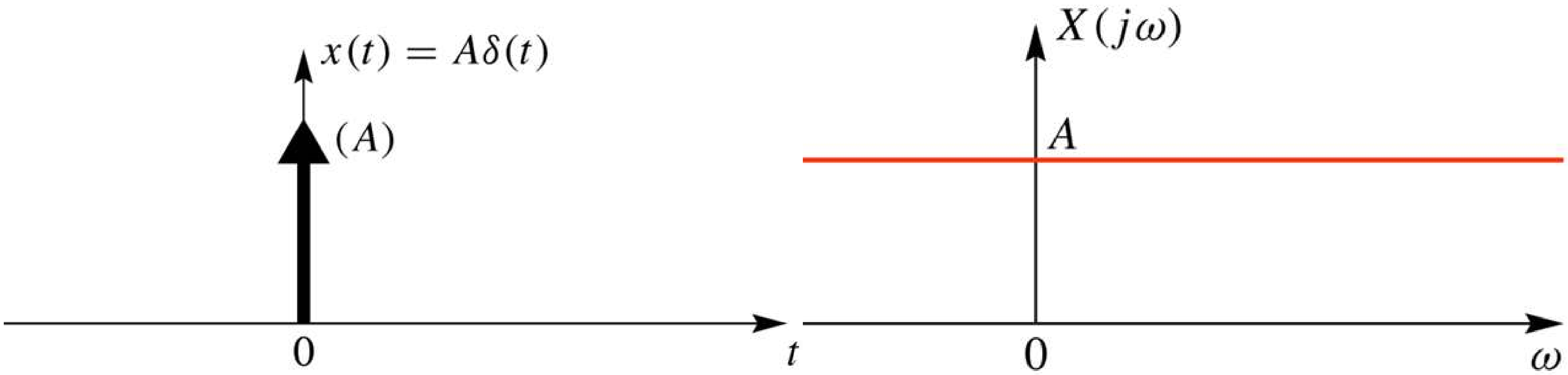
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} 1 e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\left.\frac{e^{j\omega t}}{jt}\right|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi}\frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt} = \frac{\sin(\omega_b t)}{\pi t}$$

Examples of Fourier Transform Pairs (4/6)

(4) Impulse in Time or Frequency

$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$



$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

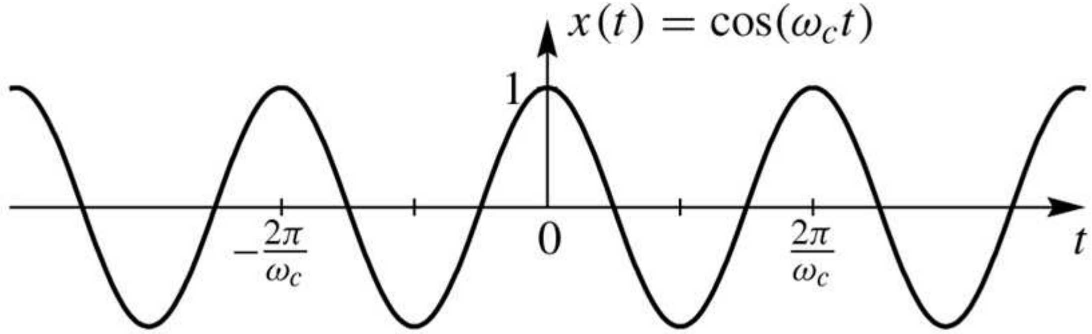
Examples of Fourier Transform Pairs (5/6)

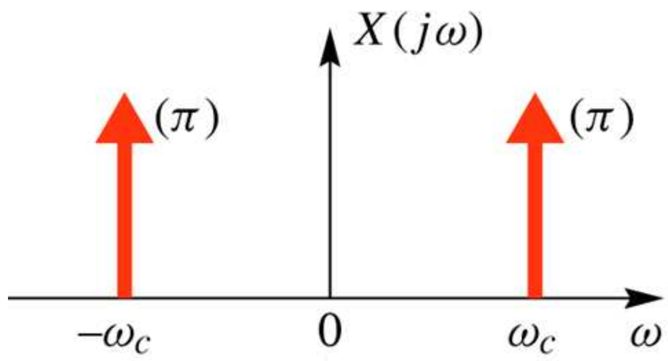
(5) Sinusoids

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$





$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

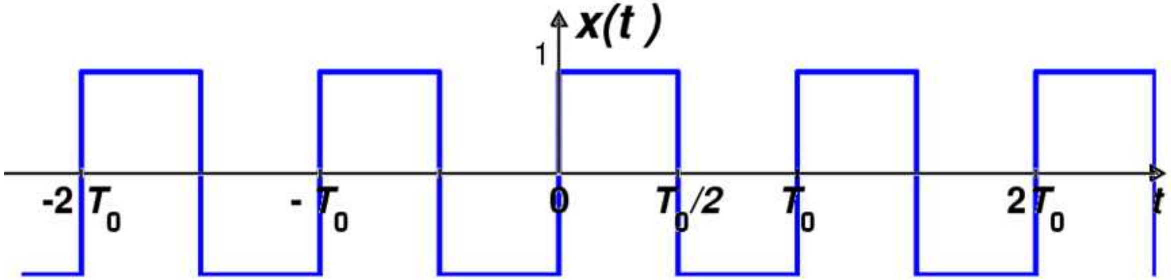
Examples of Fourier Transform Pairs (6/6)

(6) Periodic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$\text{since } e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$$



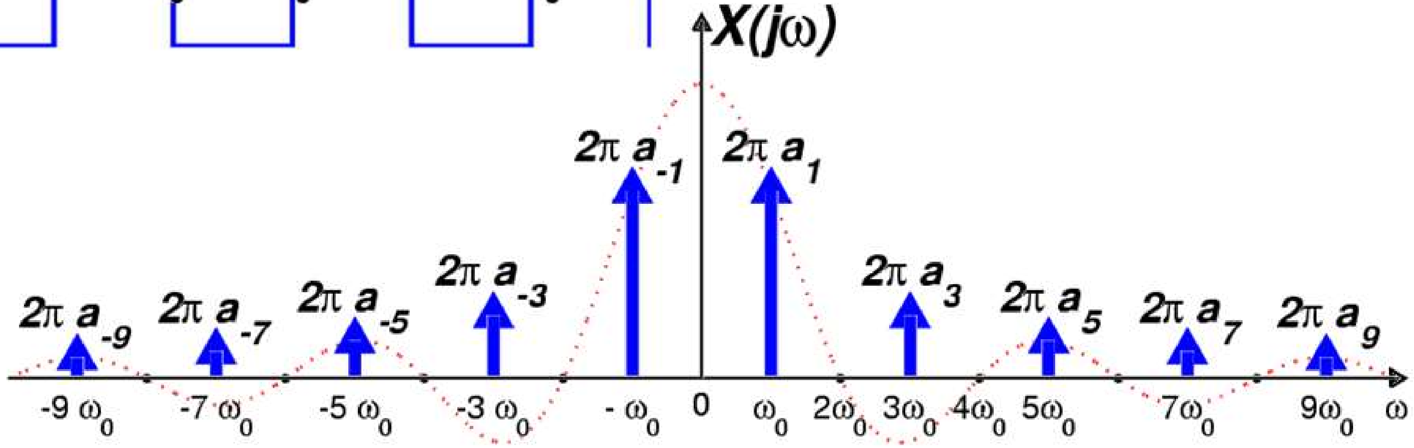


Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \quad \Leftrightarrow \quad X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$x(t) = \delta(t - t_0) \quad \Leftrightarrow \quad X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_c t} \quad \Leftrightarrow \quad X(j\omega) = 2\pi\delta(\omega - \omega_c)$$

Properties of Fourier Transform Pairs (1/9)

(1) Scaling Property

- Stretching a time signal will compress its Fourier transform.
- Compressing a time signal will stretch its Fourier transform.

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$

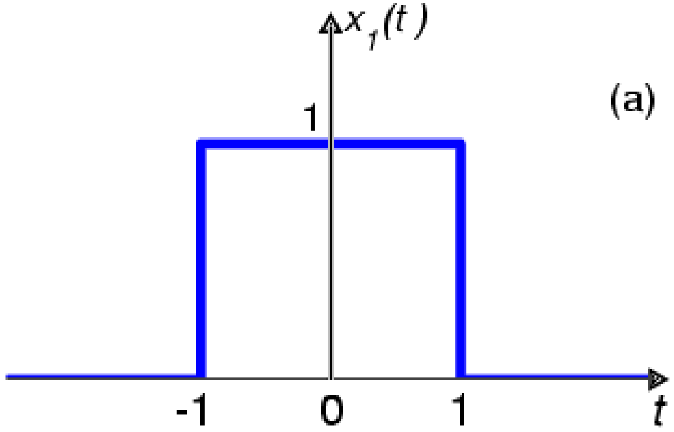
$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\lambda)e^{-j\omega(\lambda/a)}\frac{d\lambda}{|a|}$$

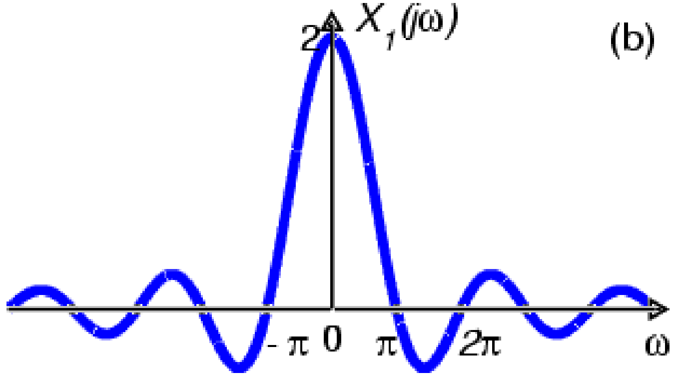
$$= \frac{1}{|a|} X(j\frac{\omega}{a})$$

- Try to make x(t) shorter, then X(jω) will get wider
- Try to make X(jω) narrower, then x(t) will have longer duration
- Cannot simultaneously reduce time duration and bandwidth

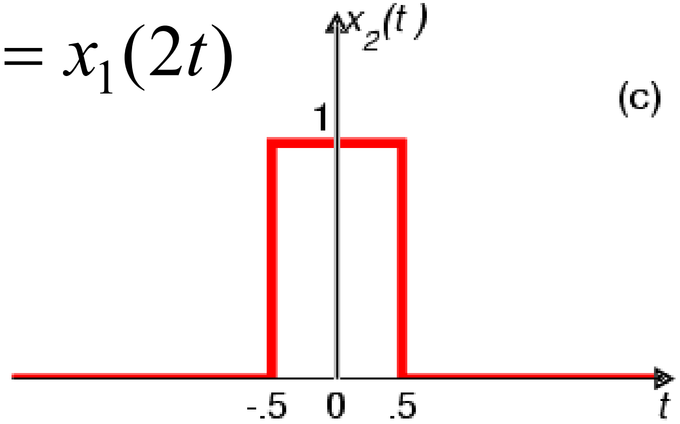
Properties of Fourier Transform Pairs (2/9)

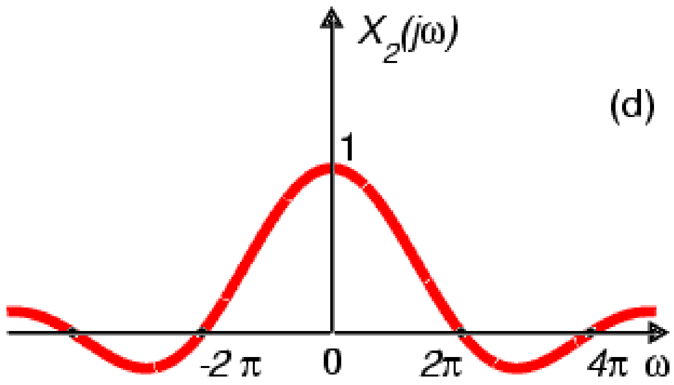
$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$





$$x_2(t) = x_1(2t)$$





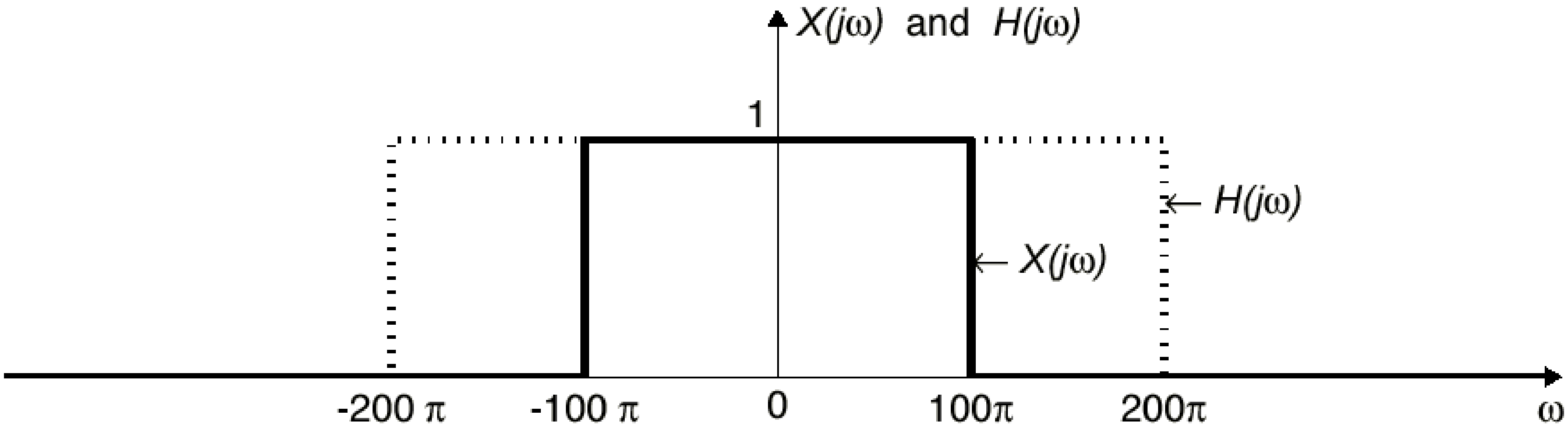
Properties of Fourier Transform Pairs (3/9)

(2) Convolution Property

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

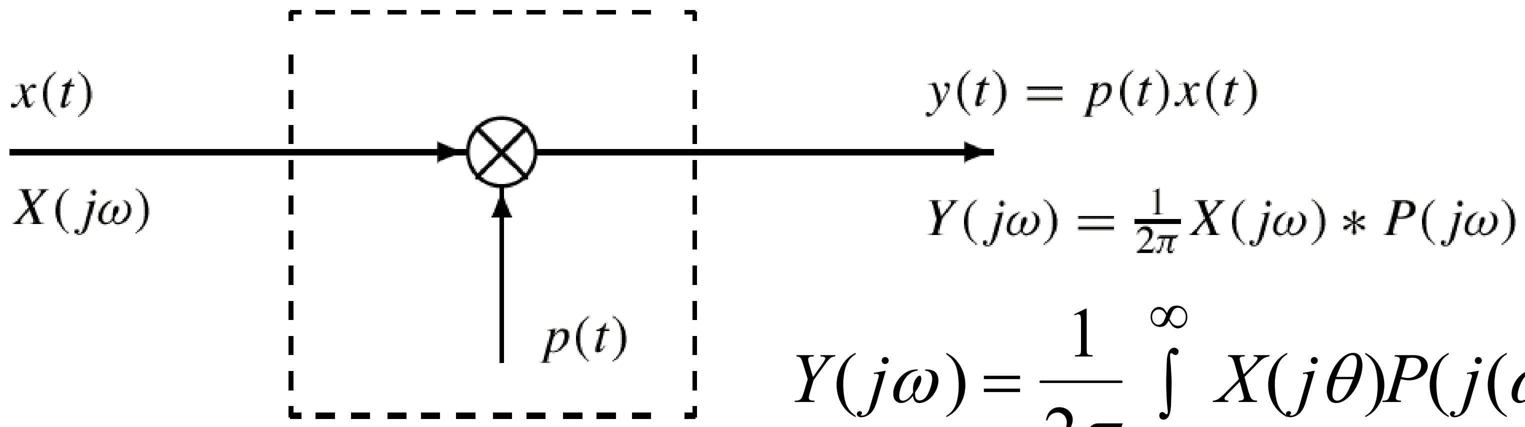
$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Properties of Fourier Transform Pairs (4/9)

(3) Multiplication Property (Modulator)

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$



$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$$

– Convolution of time functions corresponds multiplication of Fourier transforms.

– Multiplication of time functions corresponds convolution of Fourier transforms.

Properties of Fourier Transform Pairs (5/9)

- Modulation

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \quad \Leftrightarrow \quad X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$

$$p(t)=\cos(\omega_0 t) \Leftrightarrow P(j\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

$$y(t)=x(t)\cos(\omega_0 t) \Leftrightarrow$$

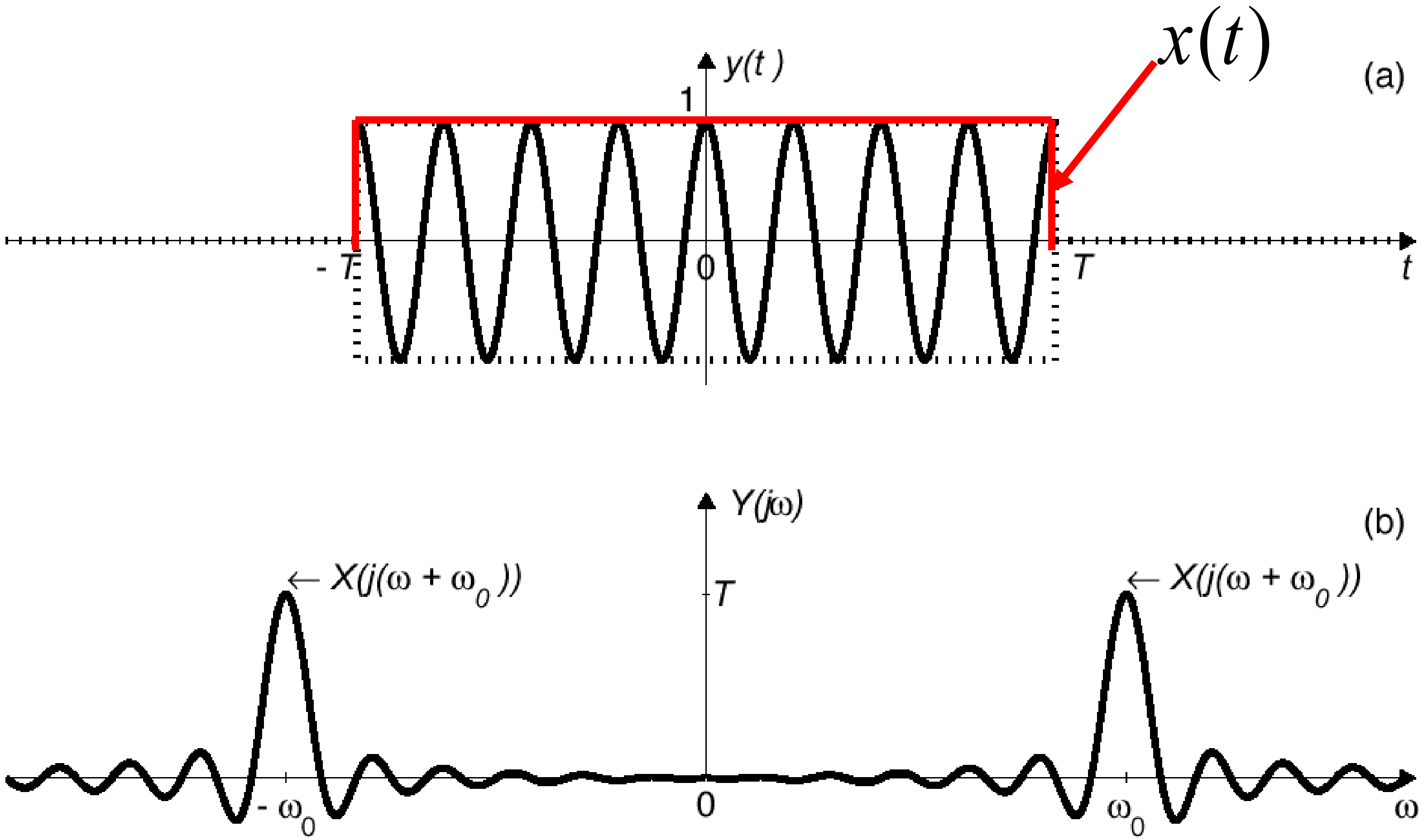
$$Y(j\omega)=\frac{1}{2\pi}X(j\omega)*[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]$$

$$=\frac{1}{2}X(j(\omega-\omega_0))+\frac{1}{2}X(j(\omega+\omega_0))$$

Properties of Fourier Transform Pairs (6/9)

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$$



Properties of Fourier Transform Pairs (7/9)

(4) Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(j(\omega - \omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi t} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}$$

Properties of Fourier Transform Pairs (8/9)

(5) Differentiation Property

$$\delta^{(1)}(t) \Leftrightarrow j\omega$$

$$\frac{d^k x(t)}{dt^k} \Leftrightarrow (j\omega)^k X(j\omega)$$

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) X(j\omega) e^{j\omega t} d\omega$$

Properties of Fourier Transform Pairs (9/9)

(6) Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

$$\int_{-\infty}^{\infty} x(t - t_d) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_d)} d\tau$$

$$= e^{-j\omega t_d} X(j\omega)$$

For example, $e^{-a(t-5)} u(t-5) \Leftrightarrow \dfrac{e^{-j\omega 5}}{a + j\omega}$

Partial Fraction Expansions

- Determine y(t) = h(t)*x(t), where

$$x(t) = \delta(t) - e^{-t}u(t) \;\text{ and }\; h(t) = e^{-2t}u(t)$$

$$X(j\omega) = 1 - \frac{1}{1+j\omega} = \frac{j\omega}{1+j\omega},\;\; H(j\omega) = \frac{1}{2+j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \left(\frac{j\omega}{1+j\omega}\right)\left(\frac{1}{2+j\omega}\right) = \frac{j\omega}{(1+j\omega)(2+j\omega)}$$

$$= \frac{A}{1+j\omega} + \frac{B}{2+j\omega} \quad \text{where } A = -1,\; B = 2$$

$$y(t) = -e^{-t}u(t) + 2e^{-2t}u(t)$$

Table of Fourier Transform Properties and Pairs

- Linearity $$ax_1(t)+bx_2(t) \Leftrightarrow aX_1(j\omega)+bX_2(j\omega)$$
- Scaling $$x(at) \Leftrightarrow \frac{1}{|a|}X(j(\tfrac{\omega}{a}))$$
- Convolution $$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$
- Multiplication $$x(t)p(t) \Leftrightarrow \frac{1}{2\pi}X(j\omega)*P(j\omega)$$
- Freq. Shifting $$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$
- Differentiation $$\frac{d^k x(t)}{dt^k} \Leftrightarrow (j\omega)^k X(j\omega)$$
- Delay $$x(t-t_d) \Leftrightarrow e^{-j\omega t_d}X(j\omega)$$

Summary

- This lecture introduced the continuous-time Fourier transform and derived a number of its properties.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \Leftrightarrow \quad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Fourier Transforms
  - Right-sided Real Exponential Signals
  - Rectangular Pulse Signals, Bandlimited Signals
  - Impulse in Time or Frequency
  - Sinusoids
  - Periodic Signal
- Properties of Fourier Transform Pairs
  - Scaling, Convolution, Multiplication (Modulator),
  - Frequency Shifting, Differentiation, Delay